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Adjoints of Sums of M-Accretive Operators and Applications to Non-Autonomous Evolutionary Equations

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Abstract: We provide certain compatibility conditions for m-accretive operators such that the adjoint of the sum is given by the closure of the sum of the respective adjoint. We revisit the proof of well-posedness of the abstract class of partial differential-algebraic equations known as evolutionary equations. We show that the general mechanism provided here can be applied to establish well-posedness for non-autonomous evolutionary equations with L_∞-coefficients thus not only generalising known results but opening up new directions other methods such as evolution families have a hard time to come by.

Keywords: Evolutionary Equations, Non-autonomous Equations, m-accretive Operators

AMS Subject Classification (2020): 35P05 (Primary), 47D99, 35Q61, 35Q59, 35K05, 35L05 (Secondary)

1 Introduction

Evolutionary Equations as introduced in the seminal paper [11] provide a Hilbert space perspective towards numerous (both linear and non-linear) time-dependent phenomena in mathematical physics. We refer to the monographs [12, 18] for a set of examples as well as further development of the theory. It is instrumental for the success of the theory of evolutionary equations that many (if not all) equations from mathematical physics can be written as a time-dependent partial differential-algebraic equation. Then, establishing the time-derivative as an m-accerive, normal operator in some weighted Hilbert space and gathering all the other unbounded operators (i.e., spatial derivative operators) in an abstract m-accretive operator A defined on some Hilbert space encoding the spatial variables, one can write evolutionary equation as an operator equation in the following form

$$(\partial_0 \mathscr{M}_0 + \mathscr{M}_1 + A)U = F, \tag{1}$$

where ∂_0 is the time-derivative, U is the unknown, F models external forces and $\mathscr{M}_0$ and $\mathscr{M}_1$ are linear operators in the considered space-time Hilbert space, which is a tensor product Hilbert space putting together temporal and spatial variables. Any standard solution theory for evolutionary equations of the form (1) provides conditions on the so-called material law operators, $\mathscr{M}_0$ and $\mathscr{M}_1$, so that

$$\overline{(\partial_0\mathscr{M}_0+\mathscr{M}_1+A)}-c$$

becomes m-accretive in the space-time Hilbert space. Quickly recall that an operator T in some Hilbert space is **accretive**[1], if for all $u\in\operatorname{dom}(T)$,

$$\langle u,Tu\rangle\geq 0.$$

T is **m-accretive**, if T is accretive and $T+\lambda$ is onto for all $\lambda>0$ (or equivalently for some $\lambda>0$). Looking into the different proofs under the various assumptions on the material law operators, one realises that the principal mechanism of showing m-accretivity is based on the following general well-known fact.

Theorem 1.1 (see also [5, Chapter 3, Theorem 1.43])**.** *Let H be a Hilbert space and $T\colon \operatorname{dom}(T)\subseteq H\to H$ a densely defined and closed linear operator. Then the following conditions are equivalent:*

(i) T is m-accretive;
(ii) T and T^* are accretive.

If $T-c$ is m-accretive for some $c>0$, then $0\in\rho(T)$.

In order to apply the last theorem showing m-accretivity of $T:=\overline{(\partial_0\mathscr{M}_0+\mathscr{M}_1+A)}-c$, it is necessary to work out the adjoint of T, which, applying standard results, boils down to computing the adjoint of $(\partial_0\mathscr{M}_0+\mathscr{M}_1+A)$ in the space-time Hilbert space. Since both $\partial_0\mathscr{M}_0$ and A are generally speaking unbounded operators, this is a non-trivial task. Our general abstract theorem provides conditions as to when for two densely defined, possibly unbounded operators S and V, we have $(S+V)^*=\overline{S^*+V^*}$. This question has been addressed for instance in [8] and the references therein. In [8] criteria are provided to ensure $(S+V)^*=S^*+V^*$, which in the case of the applications in the present manuscript cannot be used since the right-hand side is (almost never) closed. Moreover, a related question is whether the sum of two unbounded operators (not necessarily adjoints) is closable and whether one can – in one way or another – obtain an expression for this closure. Questions in this range have been posed and answered in the seminal papers by [16] and [3]; we also refer to [17]. In these papers also conditions for the invertibility of the operator sum are provided. The tools and results are developed for the general Banach space case for general sums, not necessarily of the sum of two adjoints of some given operators. Hence, the derived methods require conditions that are necessarily more involved compared to the present Hilbert space setting. For instance, note that closability alone for S^*+V^* in the present Hilbert space setting is equivalent to $\operatorname{dom}(S)\cap\operatorname{dom}(V)$ being dense.

In any case, once a formula of the type

$$(\partial_0\mathscr{M}_0+\mathscr{M}_1+A)^*=\overline{\left((\partial_0\mathscr{M}_0+\mathscr{M}_1)^*+A^*\right)}$$

is established, the accretivity of T (*and* of T^*) follow from m-accretivity of $\partial_0\mathscr{M}_0+\mathscr{M}_1$. Thus, conditions for (1) being well-posed need to address the two facts: computing the adjoint in the way sketched above needs to be possible and the problem needs to be m-accretive if $A=0$.

The aim of this article is to understand the situation for the case $\mathscr{M}_0=M_0(m_0)$ and $\mathscr{M}_1=M_1(m_0)$ are multiplication operators of multiplying in the time-variable by $t\mapsto M_0(t)$ and $t\mapsto M_1(t)$, respectively. It is known that Lipschitz continuous M_0 allows for computing the adjoint as above and suitable positive definiteness conditions for M_0 together with its (a.e. existing) derivative M_0' and M_1 lead to m-accretivity for the case $A=0$, see [14] or [18, Chapter 16].

[1] We assume every Hilbert space to be real.

A particular instance of the perspective of using operator sums to understand partial differential equations has been provided (at least) as early as [16]. However, the methods fail to apply in a straightforward manner as the coefficient M_0 is allowed to have a non-trivial kernel here (at least in the case of Lipschitz continuous M_0). We illustrate our findings in the non Lipschitz case by means of an example later on; note that this particular instance was addressed in [2]. Even though we were not able to fully rectify the arguments mentioned in this reference, their major application is concerned with continuous-in-time coefficients anyway. It seems that this condition is crucial for evolution families to be applicable. Thus, in case of non-Lipschitz continuous M_0, we establish well-posedness for a system of equations other methods (such as evolution families or evolution semigroups, see [4, Chapter VI, Section 9] and [9]) are structurally deemed to fail.

The next section is concerned with some functional analytic preliminaries, which we shall find useful in the subsequent parts. The subsequent section contains our main result concerning operator sums of m-accretive operators. The second to last section deals with applications to evolutionary equations. We summarise our findings and open problems in the conclusion section.

2 Preliminaries

Throughout this section, let H_0, H_1, H_2 be Hilbert spaces.

Lemma 2.1. *Let $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_1$ be a closed linear operator and $B\colon H_1 \to H_2$ bounded and linear. Then, if B is one-to-one and has closed range, BT is closed.*

Proof. The closed graph theorem yields that the adstriction $\tilde{B}\colon H_1 \to \operatorname{ran}(B)$ of B is a continuously invertible operator. Next, let $(x_n)_n$ be in $\operatorname{dom}(BT)$ such that $x_n \to x$ and $BTx_n \to y$ in H_0 and H_2 as $n \to \infty$ for some $x \in H_0$ and $y \in H_2$, respectively. Then, by the continuity of $(\tilde{B})^{-1}$, we infer $Tx_n = \left(\tilde{B}\right)^{-1} BTx_n \to \left(\tilde{B}\right)^{-1} y$ in H_1 as $n \to \infty$. By the closedness of T, we obtain $x \in \operatorname{dom}(T) \subseteq \operatorname{dom}(BT)$ and $Tx = \left(\tilde{B}\right)^{-1} y$. Applying $\tilde{B}$ to both sides of the latter equality, we infer $y = BTx$ as desired. □

Theorem 2.2. *Let $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_1$ be closed, $B\colon H_2 \to H_0$ be a bounded linear operator. Then, as an identity of relations, we have*

$$(TB)^* = \overline{B^*T^*}.$$

Moreover, TB is densely defined if and only if B^*T^* is a closable operator. If, B^* is one-to-one and has closed range and T is densely defined, then $B^*T^*$is closed. In particular, in this case, we have TB is densely defined and

$$(TB)^* = B^*T^*.$$

Proof. The first statement is a consequence of [18, Theorem 2.3.4]. The stated equivalence follows from $(TB)^* = \overline{B^*T^*}$ in conjunction with [18, Lemma 2.2.7]. Finally, the statement containing B^*T^* closed is a direct consequence of Lemma 2.1. □

Remark 2.3. *The assumptions on B^* are equivalent to B being onto. Indeed, by the closed range theorem, the closed range of B is then inherited by B^* and the equation $H_0 = \ker(B^*) \oplus \overline{\operatorname{ran}}(B)$ yields that (almost) surjectivity of B is equivalent to injectivity of B^*.*

Theorem 2.4. *Let $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ be densely defined and closed, $B\colon H_0 \to H_0$ be bounded, linear operator mapping onto H_0. Then*

$$(B^*TB)^* = B^*T^*B.$$

Proof. By Lemma 2.1 and Remark 2.3, B^*T is closed. It is – trivially – densely defined as so is T. Hence, by Theorem 2.2, we deduce

$$((B^*T)B)^* = \overline{B^*(B^*T)^*}.$$

Next, since $(B^*T)^*$ is a closed linear operator as the adjoint of a densely defined operator and as B^* is one-to-one with closed range (see Remark 2.3), $B^*(B^*T)^*$ is closed by Lemma 2.1; i.e., $\overline{B^*(B^*T)^*} = B^*(B^*T)^*$. Next, we compute $(B^*T)^*$. For this, using Theorem 2.2 again, we deduce

$$(T^*B)^* = \overline{B^*T^{**}} = \overline{B^*\overline{T}} = \overline{B^*T} = B^*T,$$

as B^*T is closed, again by Lemma 2.1. Computing adjoints on both sides, we infer

$$T^*B = (T^*B)^{**} = (B^*T)^*,$$

where we used that T^*B is closed. As a consequence of the above, we get

$$((B^*T)B)^* = B^*(B^*T)^* = B^*T^*B.$$ □

Proposition 2.5. *Let $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ be densely defined and closed and $B\colon H_0 \to H_0$ be a topological isomorphism. If $D \subseteq \operatorname{dom}(T)$ is a core for T (i.e., $\overline{T|_D} = T$), then $B^{-1}[D]$ is a core for B^*TB.*

Proof. B being a topological isomorphism, B^{-1} maps dense sets onto dense sets; thus $B^{-1}[D]$ is dense in H_0. Also it is elementary to see that $B^{-1}[D] \subseteq \operatorname{dom}(B^*TB)$. Finally, let $x \in \operatorname{dom}(B^*TB) = \operatorname{dom}(TB)$. Then $Bx \in \operatorname{dom}(T)$. By assumption, we find $(y_n)_n$ in D such that $y_n \to Bx$ and $Ty_n \to TBx$ in H_0 as $n \to \infty$. Defining $x_n := B^{-1}y_n \in B^{-1}[D]$, we get $x_n \to B^{-1}Bx = x$ and $TBx_n = Ty_n \to TBx$ as $n \to \infty$. The continuity of B^* yields the assertion. □

3 Adjoints of Sums of m-accretive Operators

This section is devoted to computing the adjoint of a sum of two (unbounded) operators T and S both densely defined and closed on a Hilbert space H. The aim is to provide conditions so that

$$(S+T)^* = \overline{S^* + T^*}.$$

We refer to [13], where several conditions for this equality were given. In a Hilbert space H, $(x_n)_n$ in H is said to **converge weakly** to some $x \in H$, $x_n \rightharpoonup x$, if for all $\phi \in H$, $\langle x_n, \phi\rangle \to \langle x, \phi\rangle$. For a family $(R_\varepsilon)_{\varepsilon>0}$ of bounded linear operators from a Hilbert spaces H_0 into H_1, we say $(R_\varepsilon)_{\varepsilon>0}$ **converges in the weak operator topology** to some bounded linear operator $T\colon H_0 \to H_1$, if, for all $\phi \in H_0$, $R_\varepsilon\phi \rightharpoonup T\phi$ as $\varepsilon \to 0$. The main theorem, which in turn is relevant to the applications we have in mind, reads as follows:

Theorem 3.1. *Let $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_1$ and $S\colon \operatorname{dom}(S) \subseteq H_0 \to H_1$ be two densely defined closed operators such that $\operatorname{dom}(S) \cap \operatorname{dom}(T)$ is dense. Moreover, we assume that there exist families $(L_\varepsilon)_{\varepsilon>0}$ $(K_\varepsilon)_{\varepsilon>0}$ and $(\tilde{K}_\varepsilon)_{\varepsilon>0}$ in $L(H_1)$ and $(R_\varepsilon)_{\varepsilon>0}$, in $L(H_0)$ such that $L_\varepsilon \to 1_{H_1}$, $R_\varepsilon \to 1_{H_0}$ and $K_\varepsilon, \tilde{K}_\varepsilon \to 0$ in the weak operator topology as $\varepsilon \to 0$. Moreover, assume that*

$$\begin{aligned} L_\varepsilon S &\subseteq SR_\varepsilon + K_\varepsilon, \\ L_\varepsilon T &\subseteq TR_\varepsilon + \tilde{K}_\varepsilon \end{aligned} \tag{2}$$

and

$$L_\varepsilon^*[\operatorname{dom}((S+T)^*)] \subseteq \operatorname{dom}(S^*) \cap \operatorname{dom}(T^*). \tag{3}$$

Then

$$(S+T)^* = \overline{S^* + T^*}.$$

Proof. Since in general $\overline{S^* + T^*} \subseteq (S+T)^*$ (see [18, Theorem 2.3.2]) it suffices to prove the remaining inclusion. So, let $u \in \operatorname{dom}(S+T)^*$ and set $u_\varepsilon := L_\varepsilon^* u \in \operatorname{dom}(S^*) \cap \operatorname{dom}(T^*)$. Let $v \in \operatorname{dom}(S) \cap \operatorname{dom}(T)$. Then

$$\begin{aligned}\langle (S^* + T^*)u_\varepsilon, v\rangle &= \langle L_\varepsilon^* u, (S+T)v\rangle \\ &= \langle u, L_\varepsilon(S+T)v\rangle \\ &= \langle u, (S+T)R_\varepsilon v + (K_\varepsilon + \tilde{K}_\varepsilon)v\rangle \\ &= \langle \left(R_\varepsilon^*(S+T)^* + (K_\varepsilon + \tilde{K}_\varepsilon)^*\right) u, v\rangle.\end{aligned}$$

Since $\operatorname{dom}(S) \cap \operatorname{dom}(T)$ is dense, we thus infer

$$(S^* + T^*)u_\varepsilon = \left(R_\varepsilon^*(S+T)^* + (K_\varepsilon + \tilde{K}_\varepsilon)^*\right) u \rightharpoonup (S+T)^* u,$$

where we have used $R_\varepsilon^* \to 1_{H_0}$ and $(K_\varepsilon + \tilde{K}_\varepsilon)^* \to 0$ in the weak operator topology. Since also $u_\varepsilon \rightharpoonup u$ (use again $L_\varepsilon^* \to 1_{H_1}$ in the weak operator topology), we infer that $u \in \operatorname{dom}\left(\overline{S^* + T^*}\right)$ with

$$\left(\overline{S^* + T^*}\right) u = (S+T)^* u. \qquad \square$$

Remark 3.2. *If SR_ε is bounded and $\operatorname{dom}(S) \cap \operatorname{dom}(T)$ is a core for T in the above theorem, then (3) holds true. Indeed, from (2) we infer*

$$(SR_\varepsilon + K_\varepsilon)^* \subseteq (L_\varepsilon S)^* = S^* L_\varepsilon^*.$$

If now SR_ε is bounded, the operator on the left-hand side in the above inclusion is bounded, and hence, the operator on the right-hand side is defined on H_1, meaning that $\operatorname{ran}(L_\varepsilon^*) \subseteq \operatorname{dom}(S^*)$. Hence, for $u \in \operatorname{dom}(T) \cap \operatorname{dom}(S)$ and $v \in \operatorname{dom}((S+T)^*)$ we get

$$\begin{aligned}\langle Tu, L_\varepsilon^* v\rangle &= \langle (S+T)u - Su, L_\varepsilon^* v\rangle \\ &= \langle L_\varepsilon(S+T)u, v\rangle - \langle u, S^* L_\varepsilon^* v\rangle \\ &= \langle (S+T)R_\varepsilon u + (K_\varepsilon + \tilde{K}_\varepsilon)u, v\rangle - \langle u, S^* L_\varepsilon^* v\rangle \\ &= \langle u, R_\varepsilon^*(S+T)^* v + (K_\varepsilon + \tilde{K}_\varepsilon)^* v - S^* L_\varepsilon^* v\rangle\end{aligned}$$

and since $\operatorname{dom}(T) \cap \operatorname{dom}(S)$ is a core for T, we infer that also $L_\varepsilon^* v \in \operatorname{dom}(T^*)$.

Theorem 3.3. *Let $S\colon \operatorname{dom}(S) \subseteq H \to H$ and $T\colon \operatorname{dom}(T) \subseteq H \to H$ both m-accretive. If $(1+T)^{-1}(1+S)^{-1} = (1+S)^{-1}(1+T)^{-1}$, then $\operatorname{dom}(S) \cap \operatorname{dom}(T)$ is dense in H and*

$$(S+T)^* = \overline{S^* + T^*}.$$

Remark 3.4. *Note that the conditions of Theorem 3.3 only provide a sample set of conditions sufficient for providing an example case for Theorem 3.1. In fact, some version of Theorem 3.3, which is a weaker variant than Theorem 3.1 has been used to provide a well-posedness result for (other) evolutionary equations, see [10, Lemma 1.1]. In fact, a bounded commutator assumption for S and T as well as the condition that $\varepsilon \mapsto (1+\varepsilon T)^{-1}$ and $\varepsilon \mapsto (1+\varepsilon S)^{-1}$ define uniformly bounded families of bounded linear operators on some neighbourhood of 0 is sufficient. In order to have a result readily applicable to the situation interesting for us in the following, we have opted to present a less general application of Theorem 3.1 (the most general situation provided here, is covered by Theorem 3.1 anyway).*

Before we prove Theorem 3.3, we draw some elementary consequences of the commutator condition. A first consequence of this will be that $S+T$ is densely defined.

Proposition 3.5. *Under the conditions of Theorem 3.3, the following holds:*

(i) For all $\varepsilon > 0$, we have

$$(1+\varepsilon S)^{-1}T \subseteq T(1+\varepsilon S)^{-1}.$$

(ii) $\mathrm{dom}(S)\cap\mathrm{dom}(T)$ *is a core for T. In particular,* $\mathrm{dom}(S)\cap\mathrm{dom}(T)$ *is dense in H.*

Proof. For the first statement, we observe that $(1+T)^{-1}(1+S)^{-1} = (1+S)^{-1}(1+T)^{-1}$ yields

$$(1+S)^{-1}(1+T) \subseteq (T+1)(1+S)^{-1}.$$

As a consequence,

$$(1+S)^{-1}T \subseteq T(1+S)^{-1}.$$

By [18, Lemma 9.3.3 (a)], we deduce for all $\varepsilon > 0$ that

$$(1+\varepsilon S)^{-1}T = \frac{1}{\varepsilon}(\frac{1}{\varepsilon}+S)^{-1}T \subseteq T\frac{1}{\varepsilon}(\frac{1}{\varepsilon}+S)^{-1} = T(1+\varepsilon S)^{-1}.$$

The second statement is based on the observation that $(1+\varepsilon S)^{-1} \to 1$ as $\varepsilon \to 0+$ in the strong operator topology (this follows from the strong convergence on $\mathrm{dom}(S)$ and the uniform boundedness of the resolvents). Let now $x \in \mathrm{dom}(T)$ and define $x_\varepsilon := (1+\varepsilon S)^{-1}x$. Then, $x_\varepsilon \to x$ as $\varepsilon \to 0+$ and by part 1 of the present proposition, we deduce $x_\varepsilon \in \mathrm{dom}(T)$ and

$$Tx_\varepsilon = (1+\varepsilon S)^{-1}Tx \to Tx,$$

yielding that $\mathrm{dom}(T)\cap\mathrm{dom}(S)$ is a core for T. □

Proof of Theorem 3.3. We apply Theorem 3.1. By Proposition 3.5 we see that for $L_\varepsilon := R_\varepsilon := (1+\varepsilon S)^{-1}$ and $K_\varepsilon = \tilde{K}_\varepsilon = 0$ the relations (2) are satisfied. Furthermore, by Proposition 3.5 we have that $\mathrm{dom}(S)\cap\mathrm{dom}(T)$ is dense and a core for T. Since clearly SR_ε is bounded, Remark 3.2 gives that also (3) is satisfied. Thus, the assertion follows from Theorem 3.1. □

4 Applications to Evolutionary Equations

This section is devoted to apply the previous findings to operator equations in weighted, vector-valued L_2-type spaces. The general setting can be found in [12, 18]. Throughout, let H be a Hilbert space and for $\rho \in \mathbb{R}$ we let

$$L_{2,\rho}(\mathbb{R};H) := \{f \in L_{2,\mathrm{loc}}(\mathbb{R};H);\int_{\mathbb{R}} \|f(t)\|_H^2 \exp(-2\rho t)dt < \infty\},$$

endowed with the obvious norm and corresponding scalar product. We define

$$\partial_0 \colon H^1_\rho(\mathbb{R};H) \subseteq L_{2,\rho}(\mathbb{R};H) \to L_{2,\rho}(\mathbb{R};H), \phi \mapsto \phi'.$$

For $\rho > 0$, it can be shown that ∂_0 is m-accretive with $\Re\partial_0 = \rho$.

For a bounded, strongly measurable, operator-valued function $M\colon \mathbb{R} \to L(H)$, we denote by

$$M(m_0) \in L(L_{2,\rho}(\mathbb{R};H))$$

the associated multiplication operator of multiplying by M. In applications, M will be induced by scalar-valued measurable functions; that is, $M \in L_\infty(\mathbb{R})$. We will work under the following standing hypothesis.

Hypothesis 4.1. *(i) Let $A\colon$ $\mathrm{dom}(A) \subseteq H \to H$ be m-accretive.*

(ii) *Let* $M_0, M_1 : \mathbb{R} \to L(H)$ *be strongly measurable and uniformly bounded.*
(iii) $M_0(m_0)^* = M_0(m_0)$.

Here we have employed the custom to re-use the notation A for the (canonically) extended operator defined on $L_{2,\rho}(\mathbb{R};H)$ with domain $L_{2,\rho}(\mathbb{R};\operatorname{dom}(A))$. The aim is to study the well-posedness of non-autonomous problems of the form

$$(\partial_0 M_0(m_0) + M_1(m_0) + A)U = F \tag{4}$$

under suitable commutator conditions of $M_0(m_0)$ with ∂_0 or with A.

Bounded Commutator with ∂_0

We begin to study the case when $M_0(m_0)$ and ∂_0 have a bounded commutator; that is, we assume there exists a strongly measurable and uniformly bounded mapping $M_0' : \mathbb{R} \to L(H)$ such that

$$M_0(m_0)\partial_0 \subseteq \partial M_0(m_0) - M_0'(m_0). \tag{5}$$

Remark 4.2. *In [14] it was shown that this assumption is equivalent to the Lipschitz-continuity of* M_0*. In this case,* M_0 *is differentiable almost everywhere and* M_0' *is just the so-defined derivative of* M_0*.*

Moreover, we impose the following accretivity condition on $M_0(m_0)$ and $M_1(m_0)$:

$$\exists c > 0,\, \rho_0 > 0 \,\forall \rho \geq \rho_0 : \rho M_0(t) + \frac{1}{2}M_0'(t) + M_1(t) \geq c \quad (t \in \mathbb{R} \text{ a.e.}). \tag{6}$$

Lemma 4.3. *Assume Hypothesis 4.1 together with (5) and (6). Then for each* $\rho \geq \rho_0$ *the operator*

$$\partial_0 M_0(m_0) + M_1(m_0) - c$$

is accretive. Moreover, $\operatorname{dom}(\partial_0)$ *is a core for this operator.*

Proof. If $u \in \operatorname{dom}(\partial_0)$ we infer

$$2\langle(\partial_0 M_0(m_0) + M_1(m_0))u, u\rangle = \langle(\partial_0 M_0(m_0) + M_0(m_0)\partial_0)\,u, u\rangle + \langle M_0'(m_0)u, u\rangle + 2\langle M_1(m_0)u, u\rangle.$$

Moreover, with $\partial_0^* = -\partial_0 + 2\rho$ (see [18, Corollary 3.2.6])

$$\begin{aligned}\langle \partial_0 M_0(m_0)u, u\rangle &= \langle u, M_0(m_0)\partial_0^* u\rangle \\ &= -\langle u, M_0(m_0)\partial_0 u\rangle + 2\rho\,\langle u, M_0(m_0)u\rangle,\end{aligned}$$

which gives

$$\langle(\partial_0 M_0(m_0) + M_0(m_0)\partial_0)\,u, u\rangle = 2\rho\,\langle u, M_0(m_0)u\rangle.$$

Summarising, we obtain

$$\langle(\partial_0 M_0(m_0) + M_1(m_0))u, u\rangle = \langle\left(\rho M_0(m_0) + \frac{1}{2}M_0'(m_0) + M_1(m_0)\right)u, u\rangle \geq c\|u\|^2.$$

It remains to prove that $\operatorname{dom}(\partial_0)$ is a core for $\partial_0 M_0(m_0)$. This however follows from

$$(1+\varepsilon\partial_0)^{-1}M_0(m_0) = M_0(m_0)(1+\varepsilon\partial_0)^{-1} - \varepsilon(1+\varepsilon\partial_0)^{-1}M_0'(m_0)(1+\varepsilon\partial_0)^{-1},$$

which gives

$$(1+\varepsilon\partial_0)^{-1}\partial_0 M_0(m_0) = \partial_0 M_0(m_0)(1+\varepsilon\partial_0)^{-1} - \varepsilon\partial_0(1+\varepsilon\partial_0)^{-1}M_0'(m_0)(1+\varepsilon\partial_0)^{-1}.$$

If now $u \in \operatorname{dom}(\partial_0 M_0(m_0))$ we set $u_\varepsilon := (1+\varepsilon\partial_0)^{-1}u \in \operatorname{dom}(\partial_0)$ and since $(1+\varepsilon\partial_0)^{-1} \to 1$ strongly, the latter equality proves that $u_\varepsilon \to u$ with respect to the graph norm of $\partial_0 M_0(m_0)$. □

We obtain [18, Theoerem 16.3.1] or the main result of [14] as a special case:

Theorem 4.4. *Assume Hypothesis 4.1 together with (5) and (6). Then for each $\rho \geq \rho_0$ the operator*

$$\overline{\partial_0 M_0(m_0) + M_1(m_0) + A} - c$$

is m-accretive and hence, $\overline{\partial_0 M_0(m_0) + M_1(m_0) + A}$ is boundedly invertible in $L_{2,\rho}(\mathbb{R};H)$ yielding the well-posedness of (4).

Proof. It is clear that $\partial_0 M_0(m_0) + M_1(m_0) + A - c$ is accretive as it is the sum of two accretive operators. In order to show that its closure is m-accretive, it suffices to show that its adjoint is also accretive. For this, we calculate its adjoint with the help of Theorem 3.1. We set $S := \partial_0 M_0(m_0) + M_1(m_0)$ and $T := A$. Then $C_c^\infty(\mathbb{R};\operatorname{dom}(A)) \subseteq \operatorname{dom}(S) \cap \operatorname{dom}(T)$ is dense in $L_{2,\rho}(\mathbb{R};H)$ and it is even a core for T. Setting $L_\varepsilon := (1+\varepsilon\partial_0)^{-1}$, we obtain (2) with $R_\varepsilon = L_\varepsilon$, $\tilde{K}_\varepsilon = 0$ and

$$K_\varepsilon = \varepsilon\partial_0(1+\varepsilon\partial_0)^{-1}M_0'(m_0)(1+\varepsilon\partial_0)^{-1}.$$

Finally,

$$SR_\varepsilon = \partial_0 M_0(m_0)\,(1+\varepsilon\partial_0)^{-1} = (1+\varepsilon\partial_0)^{-1}\partial_0 M_0(m_0) + K_\varepsilon$$

is bounded, and hence, (3) holds by Remark 3.2. Thus, we can apply Theorem 3.1 and obtain

$$(\partial_0 M_0(m_0) + M_1(m_0) + A)^* = \overline{(\partial_0 M_0(m_0) + M_1(m_0))^* + A^*}.$$

Since clearly A^* is accretive, it remains to prove the strict accretivity of $(\partial_0 M_0(m_0) + M_1(m_0))^* = (\partial_0 M_0(m_0))^* + M_1(m_0)^*$. In order to work out the first adjoint we recall that $\operatorname{dom}(\partial_0)$ is a core for $\partial_0 M_0(m_0)$ and hence

$$(\partial_0 M_0(m_0))^* = \left(M_0(m_0)\partial_0 + M_0'(m_0)\right)^* = \partial_0^* M_0(m_0) + M_0'(m_0).$$

Now, as in Lemma 4.3 one proves that (6) yields the accretivity of $\partial_0^* M_0(m_0) + M_0'(m_0) + M_1(m_0) - c$. □

Trivial Commutator with A

Here, we assume a commutator condition with A. To keep things simple, we assume that

$$\text{there exists } d > 0 \text{ such that } M_0(t) \geq d \text{ for almost every } t \in \mathbb{R} \tag{7}$$

and that

$$M_0(m_0)A \subseteq AM_0(m_0). \tag{8}$$

Lemma 4.5. *Assume Hypothesis 4.1 together with (7) and (8), we have*

$$M_0(m_0)A = AM_0(m_0).$$

Proof. The inclusion $M_0(m_0)A \subseteq AM_0(m_0)$ leads to

$$M_0(m_0)(A+1) \subseteq (A+1)M_0(m_0).$$

Now, the right-hand side operator is one-to-one and the left-hand side is onto. Hence,

$$M_0(m_0)(A+1) = (A+1)M_0(m_0),$$

which yields the assertion □

Remark 4.6. *It is a consequence of the definition of the square root (see also [12, Theoerem B.8.2 and its proof]) that*

$$M_0(m_0)A \subseteq AM_0(m_0)$$

leads to

$$M_0(m_0)^{1/2}A \subseteq AM_0(m_0)^{1/2};$$

thus, by Lemma 4.5, it follows

$$M_0(m_0)^{1/2}A = AM_0(m_0)^{1/2}.$$

In particular, we obtain

$$M_0(m_0)^{-1/2}A = AM_0(m_0)^{-1/2}.$$

For motivating the main result of this section, we recall the evolutionary equation from (4)

$$(\partial_0 M_0(m_0) + M_1(m_0) + A)\,U = F.$$

After multiplication by $M_0(m_0)^{1/2}$ the latter can be rewritten as

$$\begin{aligned}&\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}\big(M_1(m_0)+A\big)M_0(m_0)^{-1/2}\right)M_0(m_0)^{1/2}U\\ &= M_0(m_0)^{1/2}F.\end{aligned}$$

Using the latter remark and (8), we get

$$\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2} + A\right)M_0(m_0)^{1/2}U = M_0(m_0)^{1/2}F.$$

Remark 4.7. *If instead of the above equation, we consider*

$$(M_0(m_0)\partial_0 + M_1(m_0) + A)\,U = F,$$

we may multiply by $M_0(m_0)^{-1/2}$ instead and eventually obtain

$$\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{-1/2}M_1(m_0)M_0(m_0)^{1/2} + A\right)M_0(m_0)^{-1/2}U = M_0(m_0)^{-1/2}F,$$

being basically of the same shape of equation as the one above with $\partial_0 M_0(m_0)$.

The main result of this section is the following.

Theorem 4.8. *Assume Hypothesis 4.1 together with (7) and (8). Then*

$$\begin{aligned}&\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2} + A\right)^*\\ &= \overline{\left(M_0(m_0)^{1/2}\partial_0^* M_0(m_0)^{1/2} + M_0(m_0)^{-1/2}M_1(m_0)^*M_0(m_0)^{-1/2} + A^*\right)}.\end{aligned}$$

Proof. First of all note that $M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2}$ is a bounded linear operator and can, thus, be assumed to be 0 when computing the adjoint. Next,

$$\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2}\right)^* = M_0(m_0)^{1/2}\partial_0^* M_0(m_0)^{1/2}$$

by Theorem 2.4 applied to $T = \partial_0$ and $B = M_0(m_0)^{1/2} = B^*$. Note that it particularly follows that

$$\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2}\right)$$

is m-accretive. Thus, for proving the present theorem, it suffices to apply Theorem 3.3 to $T = M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2}$ and $S = A$. What remains is to show the commutativity of the resolvents:

$$\begin{aligned}
(1+T)^{-1}(1+S)^{-1} &= (1+M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2})^{-1}(1+A)^{-1} \\
&= (M_0(m_0)^{1/2}(M_0(m_0)^{-1}+\partial_0)M_0(m_0)^{1/2})^{-1}(1+A)^{-1} \\
&= M_0(m_0)^{-1/2}(M_0(m_0)^{-1}+\partial_0)^{-1}M_0(m_0)^{-1/2}(1+A)^{-1} \\
&= M_0(m_0)^{-1/2}(M_0(m_0)^{-1}+\partial_0)^{-1}(1+A)^{-1}M_0(m_0)^{-1/2} \\
&= M_0(m_0)^{-1/2}\left((1+A)(M_0(m_0)^{-1}+\partial_0)\right)^{-1}M_0(m_0)^{-1/2} \\
&= M_0(m_0)^{-1/2}\left((M_0(m_0)^{-1}+\partial_0)(1+A)\right)^{-1}M_0(m_0)^{-1/2} \\
&= (1+S)^{-1}(1+T)^{-1},
\end{aligned}$$

where we used $(1+A)\partial_0 = \partial_0(1+A)$ and Lemma 4.5 for $M_0(m_0)^{-1}(1+A) = (1+A)M_0(m_0)^{-1}$. Hence, Theorem 3.3 is applicable and the assertion follows. □

Lemma 4.9. *Assume Hypothesis 4.1 together with (7). Then, for all $c > 0$ there exists $\rho_0 > 0$ such that for each $\rho \geq \rho_0$ the operator*

$$M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2} - c$$

is m-accretive on $L_{2,\rho}(\mathbb{R};H)$.

Proof. For $u \in \operatorname{dom}(\partial_0 M_0(m_0)^{1/2})$ we have

$$\begin{aligned}
&\langle\left(M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2}\right)u,u\rangle \\
&= \rho\|M_0(m_0)^{1/2}u\|^2 - \|M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2}\|\|u\|^2 \\
&\geq \left(\rho d - \|M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2}\|\right)\|u\|^2.
\end{aligned}$$

Choosing now ρ large enough, we infer the strict accretivity of the operator. Since its adjoint is of the form

$$M_0(m_0)^{1/2}\partial_0^* M_0(m_0)^{1/2} + M_0(m_0)^{-1/2}M_1(m_0)^* M_0(m_0)^{1/2}$$

the same argument shows that for ρ large enough, this operator is also accretive, and hence, the assertion follows. □

Corollary 4.10. *Assume Hypothesis 4.1 together with (7) and (8). Then the operator*

$$\overline{\partial_0 M_0(m_0) + M_1(m_0) + A}$$

is boundedly invertible in $L_{2,\rho}(\mathbb{R};H)$ for $\rho > 0$ large enough.

Proof. In the present situation, consider

$$\tilde{T} := \overline{T+S},$$

where $T := M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{1/2}M_1(m_0)M_0(m_0)^{-1/2}$ and $S := A$. We will show that $\tilde{T} - c$ is m-accretive for some $c > 0$. By assumption and Lemma 4.9, it is not difficult to see that $\tilde{T} - c$ is accretive for some $c > 0$ and all large enough $\rho > 0$. Using the formula for the adjoint in Theorem 4.8 and taking into account the accretivtiy of $T^* - c$, we have that $\left(\tilde{T}\right)^* - c$ is, too, accretive. Hence, $0 \in$

$\rho(\tilde{T})$. The reformulation just before Remark 4.7 yields the assertion by multiplying $\tilde{T}$ by the topological isomorphism $M_0(m_0)^{-1/2}$ from the left and $M_0(m_0)^{1/2}$ from the right. □

Remark 4.11. *A similar result holds under the assumption that*

$$M_0(m_0)^{1/2}\partial_0 M_0(m_0)^{1/2} + M_0(m_0)^{-1/2} M_1(m_0) M_0(m_0)^{1/2} - c$$

is m-accretive for some $c > 0$. Then

$$0 \in \rho\left(\overline{M_0(m_0)\partial_0 + M_1(m_0) + A}\right).$$

Next we treat a non-autonomous example of a transport equation on a graph with finitely many edges of equal length 1. The following is merely to illustrate an example, where one can have L_∞-dependence of time-varying transport velocities in the graph. Note that well-posedness results of an autonomous version of this kind of problems are known from [6] in L_1; and [7] with spatially dependent velocities. The corresponding non-autonomous situation has been adressed in [1] with a weak differentiability condition on the time-dependent velocities. In the following example, we dispense with any regularity conditions on the velocity. However, instead we need rather strong commutativity properties for the velocity matrix with the matrix describing the boundary conditions. Note that time- and spatially dependent velocities so that the time-dependence is Lipschitz regular can also be dealt with within an evolutionary equations setting. Indeed, the perspective provided in [15] together with the (abstract) non-autonomous well-posedness result in [18, Theorem 16.3.1] or [14, Theorem 2.13] can be understood in this way. This particularly emphasises the interest of PDEs with L_∞-time dependence only.

Example 4.12. *Let $d \in \mathbb{N}$ and consider $H := L_2(0,1)^d$. Moreover, let $B \in \mathbb{R}^{d\times d}$ with $\|B\| \leq 1$ and consider the operator A given by*

$$\begin{aligned} \operatorname{dom}(A) &:= \{u \in H^1(0,1)^d\,;\, u(0) = Bu(1)\}, \\ Au &:= u'. \end{aligned}$$

Then A is m-accretive by [15, Theorem 4.1]. Moreover, let $c_1,\ldots,c_d \in L_\infty(\mathbb{R})$ such that $c_1,\ldots,c_d \geq k > 0$ almost everywhere. We set $M_0(t) := \operatorname{diag}(c_j(t))$ and assume that $BM_0(t) = M_0(t)B$ for almost every $t \in \mathbb{R}$. Then we clearly have for $u \in L_{2,\rho}(\mathbb{R};\operatorname{dom}(A))$ that $M_0(m_0)u \in L_{2,\rho}(\mathbb{R};H^1(0,1)^d)$ and that

$$BM_0(t)u(1) = M_0(t)Bu(1) = M_0(t)u(0);$$

that is, $M_0(m_0)u \in L_{2,\rho}(\mathbb{R};\operatorname{dom}(A))$. Moreover,

$$AM_0(m_0)u = M_0(m_0)u' = M_0(m_0)Au,$$

which shows $M_0(m_0)A \subseteq AM_0(m_0)$. Hence, by our findings above, the non-autonomous problem

$$(\partial_0 M_0(m_0) + A)\,u = f$$

is well-posed in $L_{2,\rho}(\mathbb{R};L_2(0,1)^d)$ if we choose ρ large enough.

5 Conclusions

We provided applicable conditions for m-accretive operators so that the adjoint of the sum can be represented as the closure of the sum of adjoints of the individual operators. We applied this observation to evolutionary equations and developed well-posedness criteria for the same. The case of the operator M_0 being L_∞-in time only and having non-trivial (possibly time-independent) kernel remains open for easily applicable conditions establishing well-posedness. Thus, to properly address the general situation of time-dependent partial differential algebraic equations appears to be a challenge for future research.

Data Availability Statement

No new data were created or analysed during this study. Data sharing is not applicable to this article.

Underlying and related material

No underlying or related material.

Author contributions

CRediT: Rainer Picard, Sascha Trostorff, Marcus Waurick: Conceptualisation, Methodology, Writing, Investigation.

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No competing interests to declare.

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