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Abstract Dissipative Hamiltonian Differential-Algebraic Equations Are Everywhere

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Abstract: In this paper we study the representation of partial differential equations (PDEs) as abstract differential-algebraic equations (DAEs) with dissipative Hamiltonian structure (adHDAEs). We show that these systems not only arise when there are constraints coming from the underlying physics, but many standard PDE models can be seen as an adHDAE on an extended state space. This reflects the fact that models often include closure relations and structural properties. We present a unifying operator theoretic approach to analyze the properties of such operator equations and illustrate this by several applications.

Keywords: Abstract Differential-Algebraic Equation, Closure Relation, Dissipative Hamiltonian System, Energy Based Modelling, Operator Pair, Regular Pair, Singular Pair

AMS Subject Classification (2020): 37l05, 37l20, 47d06, 47f05, 93b28, 93c05.

1 Introduction

In this paper we study the mathematical modeling, the analytical theory and the representation of abstract linear differential-algebraic equations (DAEs) of the form

$$\mathscr{E}\dot{x}(t) = \mathscr{A}\mathscr{Q}x(t) \tag{1}$$

on the infinite-dimensional Hilbert space $\mathbb{X}$ with inner product $\langle \cdot, \cdot \rangle$. We assume that $\mathscr{A} : D(\mathscr{A}) \subseteq \mathbb{X} \to \mathbb{X}$ is a *dissipative linear operator*, i.e., $\langle \mathscr{A}x, x\rangle + \langle x, \mathscr{A}x\rangle \leq 0$ for all x in the domain of $\mathscr{A}$. The operators $\mathscr{E} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{X} \to \mathbb{X}$ are assumed to be bounded linear operators that satisfy further geometric conditions and define an *energy functional or Hamiltonian* via

$$\mathscr{H}(x) := \langle \mathscr{E}x, \mathscr{Q}x\rangle, \tag{2}$$

which is assumed to be non-negative, i.e., $\mathscr{H}(x) \geq 0$ for all $x \in \mathbb{X}$.

We call this class of problems *abstract dissipative Hamiltonian DAEs (adHDAEs)*.

Abstract differential-algebraic systems do not only arise by including constraints coming from the underlying physical system, see e.g. [13, 16, 26], but many standard systems of partial differential equations

(PDEs) can be viewed as abstract differential-algebraic equation on an extended state-space. We present several applications where this is the case.

The class of adHDAEs is also strongly motivated by modeling physical systems in the model class of (abstract) port-Hamiltonian differential-algebraic systems (pHDAEs), a class which is of great relevance in many applications and has recently seen a huge number of applications in almost all physical domains, see e.g. [2, 3, 4, 14, 15, 18, 21, 24, 44, 32, 34, 35, 38, 45, 37]. To illustrate the concept of adHDAEs, consider the following example.

Example 1 *[23] The vibrating string in one space dimension can be modelled by the PDE*

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial \zeta}\left(T \frac{\partial w}{\partial \zeta}\right), \tag{3}$$

where ρ is the mass density, w is the vertical displacement, T is the Young modulus of the material, ζ is in a one-dimensional spatial domain, and t the time.

The port-Hamiltonian modeling approach, see [23], introduces the extended state

$$z(t) = \begin{bmatrix} \rho \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{bmatrix}$$

in the state space $\mathbb{X} = L^2(\Omega;\mathbb{R}^2)$, with Ω the spatial interval, and leads to a representation of (3) given by

$$\dot z(t) = \underbrace{\begin{bmatrix} 0 & \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \zeta} & 0 \end{bmatrix}}_{:= \quad \mathscr{A}} \underbrace{\begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix}}_{\tilde{\mathscr{Q}}} z(t) \qquad z(t),$$

with a Hamiltonian $\mathscr{H}(z(t)) = \langle z(t), \tilde{\mathscr{Q}} z(t)\rangle$. If the mass density ρ is close to zero, then it is important to analyze what happens when one considers the density $\rho = 0$. For ρ close to zero, it is more appropriate to consider a different state

$$x(t) = \begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{bmatrix},$$

which leads to a representation

$$\underbrace{\begin{bmatrix} \rho & 0 \\ 0 & 1 \end{bmatrix}}_{\mathscr{E}} \dot x(t) = \underbrace{\begin{bmatrix} 0 & \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \zeta} & 0 \end{bmatrix}}_{\mathscr{A}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}}_{\mathscr{Q}} x(t),$$

$$\mathscr{E} \qquad \dot x(t) = \qquad \mathscr{A} \qquad \mathscr{Q} \qquad x(t)$$

where we have introduced the matrices $\mathscr{E}$ and $\mathscr{Q}$ and the differential operator $\mathscr{A}$.

Note that by this change of variables the value of the Hamiltonian $\mathscr{H}$ stays the same, i.e.,

$$\mathscr{H}(t) = \langle z(t), \tilde{\mathscr{Q}} z(t)\rangle = \langle \mathscr{E} x(t), \mathscr{Q} x(t)\rangle.$$

However, in this formulation, we can set both $\rho = 0$, and $T = 0$ and then either $\mathscr{E}$ or $\mathscr{Q}$ or both become singular.

For invertible $\mathscr{E}$, we can express this system as a standard wave equation, of which it is known that it will generate a contraction semigroup, provided the appropriate boundary conditions are posed, [23].

Remark 2 *Example 1 demonstrates that the use of differential-algebraic equations is essential when considering limiting situations, see also [4, 46, 44, 39] for detailed discussions. In many applications one can resolve the constraint equations and return to explicit formulations in the time derivative. But this is not always a good mathematical formulation for several reasons. First of all it may happen that the resulting system is much more sensitive under perturbations. But more important, by resolving the constraints, they are not visible in the equations any longer, even though they usually are of physical relevance. Furthermore, they are then also not enforced during a numerical simulation of the system, see [5, 20, 25] and this can lead to a drift of the numerical solution from the constraint manifold.*

Example 1 is a motivation to study the properties of adHDAEs of the form (1) in which both operators $\mathscr{E}$ and $\mathscr{Q}$ may be *singular* matrices or non-bijective operators, and where $\mathscr{A}$ generates a contraction semigroup on the Hilbert space $\mathbb{X}$. When modelling physical systems in a modular fashion then often not only $\mathscr{E}$ and $\mathscr{Q}$ may be singular but the equation (1) may be overdetermined or not uniquely solvable. For general abstract DAEs this is hard to analyze but we present a simple characterization of singularity for (1) in Subsection 2.1.

One may have the impression that the case that $\mathscr{E}$ and/or $\mathscr{Q}$ are singular is a very special case that is not encountered often when modeling physical processes. However, we will demonstrate that this is almost the standard case. To illustrate this, consider the following example.

Example 3 *Consider the derivation of the diffusion/heat equation in a one-dimensional domain. The defining relation between the temperature T and the heat flux J is given by the PDE*

$$\frac{\partial T}{\partial t} = -\alpha \frac{\partial J}{\partial \zeta}, \tag{4}$$

where $\alpha > 0$ is the diffusivity constant. Using Fourier's law to model the heat flux as proportional (with thermal conductivity k) to the spatial derivative of the temperature, i.e.,

$$J = -k\frac{\partial T}{\partial \zeta} \tag{5}$$

gives the standard diffusion/heat equation

$$\frac{\partial T}{\partial t} = k\alpha \frac{\partial^2 T}{\partial \zeta^2}.$$

However, we can also express the system as adHDAE system

$$\underbrace{\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathscr{E}} \underbrace{\frac{\partial}{\partial t}\begin{bmatrix} T \\ J \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{\partial}{\partial \zeta} \\ -\frac{\partial}{\partial \zeta} & -1 \end{bmatrix}}_{\mathscr{A}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & k^{-1} \end{bmatrix}}_{\mathscr{Q}} \underbrace{\begin{bmatrix} T \\ J \end{bmatrix}}_{x(t)} \tag{6}$$

with $\mathbb{X}$ as in the previous example. Thus it is in the form (1) with $\mathscr{E}$ being singular. Note that in this case the singularity of $\mathscr{E}$ is not caused by a physical parameter becoming zero, but it is a direct consequence of the closure relation (5). Since these closure relations appear almost everywhere in mathematical modelling, we see that a singular $\mathscr{E}$ is very common.

The discussed examples demonstrate that in modeling with adHDAEs different representations are possible, and some are preferable to others, e.g. in the case of limiting situations.

The operator $\mathscr{A}$ in (6) is very similar to that in (3), and so one may think that their properties are related, but it is well-known that the wave and heat equation behave completely differently, the first has oscillating solution behavior while the second is diffusive, so the solution decays. However, as we will demonstrate, the solution theory of the two PDEs is strongly related, see Example 28 below.

The structure in (1) is also motivated by the class of finite dimensional dissipative Hamiltonian descriptor systems introduced in [3], see also [30, 32] that have the form (1) with $\mathscr{A} = \mathscr{J} - \mathscr{R}$, where $\mathscr{J}$ is (formally) skew-adjoint, and $\mathscr{E}^*\mathscr{Q}$ as well as $\mathscr{R}$ are self-adjoint and nonnegative (positive semidefinite).

The paper is organized as follows. In Section 2 we introduce our basic set-up together with several assumptions. In Section 3 we study the solution theory of adHDAEs of the form (1). These results are illustrated in Section 4 by several examples, showing their applicability. In these examples we also recover many results, which often were obtained by other methods. In Section 5 we treat the case in which the singularity of $\mathscr{E}$ restricts the state space, and again our result is illustrated by examples.

2 Representation of adHDAEs

In this section we study adHDAEs of the form (1) on an infinite-dimensional Hilbert space $\mathbb{X}$. In order to analyze the solution properties we make some general assumptions on the structure of $\mathscr{A}, \mathscr{E}$, and $\mathscr{Q}$.

Consider an abstract dissipative Hamiltonian differential-algebraic equation (adHDAE)

$$\mathscr{E}_{ext}\dot{x}(t) = \mathscr{A}_{ext}\mathscr{Q}_{ext}x(t) \tag{7}$$

of the form (1) with the following structural properties.

Assumption 4 *i) The state-space is a Hilbert space* $\mathbb{X}_{ext} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3$.

ii) The operator $\mathscr{A}_{ext} = \begin{bmatrix} \mathscr{A}_{1,ext} \\ \mathscr{A}_{2,ext} \\ \mathscr{A}_{3,ext} \end{bmatrix}$ *is a* dissipative operator *on* $\mathbb{X}_{ext}$*, i.e.,* $\mathrm{Re}\langle \mathscr{A}_{ext}x, x\rangle \leq 0$ *for all* x *in the domain* $D(\mathscr{A}_{ext})$ *of* $\mathscr{A}_{ext}$.

iii) The operators $\mathscr{E}_{ext}$ *and* $\mathscr{Q}_{ext}$ *are block-operators of the form*

$$\mathscr{E}_{ext} = \begin{bmatrix} \mathscr{E}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathscr{E}_3 \end{bmatrix}, \qquad \mathscr{Q}_{ext} = \begin{bmatrix} \mathscr{Q}_1 & 0 & 0 \\ 0 & \mathscr{Q}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

where $\mathscr{E}_1, \mathscr{E}_3, \mathscr{Q}_1$ *and* $\mathscr{Q}_2$ *are bounded and boundedly invertible. Furthermore, we assume that* $\mathscr{E}_1^*\mathscr{Q}_1$ *is* coercive*, i.e., it is self-adjoint and (strictly) positive.*

iv) There exists an $s \in \mathbb{C}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ *such that the operator* $s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}$ *with domain* $\{x \in \mathbb{X} \mid \mathscr{Q}_{ext}x \in D(\mathscr{A}_{ext})\}$ *is boundedly invertible.*

Remark 5 *Assumption 4 seems to be very restrictive at first sight. However, as we will demonstrate, it holds for many examples and it allows us to prove our main results. However, this assumption can be relaxed in many particular cases by using different proof techniques.*

See also [13] for the analysis of the chain-index under this assumption.

In the setting of finite dimensional DAEs, see [29], condition iv) in Assumption 4 implies that the pair $(\mathscr{E}_{ext}, \mathscr{A}_{ext}\mathscr{Q}_{ext})$ forms a *regular pair*, see e.g. [25]. We will use this terminology also in the infinite dimensional case when the operators satisfies Assumption 4. iv). A characterization when a pair is regular or singular is given in Subsection 2.1.

Remark 6 *In the case that $\mathscr{E}_{ext}, \mathscr{Q}_{ext}$ are matrices, the condition that $\mathscr{E}_{ext}^* \mathscr{Q}_{ext}$ is self-adjoint means that the columns of*

$$\begin{bmatrix} \mathscr{E}_{ext} \\ \mathscr{Q}_{ext} \end{bmatrix}$$

span an isotropic subspace of $\mathbb{X} \times \mathbb{X}^* = \mathbb{X} \times \mathbb{X}$, see e.g. [44, 39], which is a Lagrange subspace if the dimension is maximal, i.e., that of $\mathbb{X}$. This is the case if and only if the pair $(\mathscr{E}_{ext}, \mathscr{Q}_{ext})$ is regular. For Lagrange subspaces the representation (8) can always be achieved by a change of basis using a cosine-sine decomposition, see [29, 33].

Remark 7 *From the modeling point of view systems of the form (1) lead to a natural definition of an* energy functional (Hamiltonian)

$$\mathscr{H}(x) := \langle \mathscr{E}_{ext} x, \mathscr{Q}_{ext} x \rangle. \tag{9}$$

However, the definition of the Hamiltonian is by no means unique, in particular the choice of variables in the kernels of $\mathscr{E}_{ext}$ and $\mathscr{Q}_{ext}$ is arbitrary and thus there are many different representations of the state variables with the same Hamiltonian, see Example 1. Under the conditions in Assumption 4, we have that

$$\mathscr{H}(x_1) = \langle \mathscr{E}_1 x_1, \mathscr{Q}_1 x_1 \rangle = \mathscr{H}(x),$$

i.e., the Hamiltonian may also be defined on a restricted subspace.

For a detailed discussion of this topic of different representations in the finite dimensional case, see [44, 39].

Looking at a system (7) that satisfies Assumption 4, we see that the third state, x_3, does not influence the first nor the second state. However, its behaviour is dictated by the other two. So we could regard $\dot{x}_3$ in (7) as a kind of *output to the system*. Since we are mainly interested in the dynamics of the first state, a natural question is if we can find a reduced representation of the system with similar properties by removing the third state. This topic has been discussed extensively in the case of finite dimensional port-Hamiltonian DAEs, see [3, 30]. Since the conditions in Assumption 4 include $\mathscr{E}_3$ and $\mathscr{A}_{3,ext}$, it is not clear a priori whether similar properties still hold without these assumptions. Our first result shows that this is indeed the case.

Theorem 8 *Consider an adHDAE of the form (1) that satisfies Assumption 4. Introduce the operator*

$$\mathscr{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} \mathscr{A}_{1,ext} \\ \mathscr{A}_{2,ext} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \tag{10}$$

with domain

$$D(\mathscr{A}) = \{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{X}_1 \oplus \mathbb{X}_2 \mid \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in D(\mathscr{A}_{ext}) \}. \tag{11}$$

Then $\mathscr{A}$ is dissipative and with

$$\mathscr{E} = \begin{bmatrix} \mathscr{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathscr{Q} = \begin{bmatrix} \mathscr{Q}_1 & 0 \\ 0 & \mathscr{Q}_2 \end{bmatrix}, \tag{12}$$

the pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular.

Proof. For $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in D(\mathscr{A})$ we have

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathscr{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \mathscr{A}_{ext} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \right\rangle.$$

Since by assumption the last expression has non-positive real part, we see that $\mathscr{A}$ is dissipative.

Let $s \in \mathbb{C}^+$ be as in Assumption 4 iv). We will show that the operator $s\mathscr{E} - \mathscr{A}\mathscr{Q}$, with domain $\{x \in \mathbb{X}_1 \oplus \mathbb{X}_2 \mid \mathscr{Q}x \in D(\mathscr{A})\}$, is boundedly invertible.

Let $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{X}_1 \oplus \mathbb{X}_2$ be such that $\mathscr{Q}x \in D(\mathscr{A})$ and $(s\mathscr{E} - \mathscr{A}\mathscr{Q})x = 0$. Define $x_3 = \frac{1}{s}\mathscr{E}_3^{-1}\mathscr{A}_{3,ext}\left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \\ 0 \end{smallmatrix}\right]$. With this choice, then

$$(s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Since the pair $(\mathscr{E}_{ext}, \mathscr{A}_{ext}\mathscr{Q}_{ext})$ is regular, this implies, in particular, that $x_1 = 0$ and $x_2 = 0$. Thus $(s\mathscr{E} - \mathscr{A}\mathscr{Q})$ is injective.

For $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in \mathbb{X}_1 \oplus \mathbb{X}_2$, define

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = (s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext})^{-1} \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}. \tag{13}$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} := \begin{bmatrix} \mathscr{Q}_1\tilde{x}_1 \\ \mathscr{Q}_2\tilde{x}_2 \\ 0 \end{bmatrix} = \mathscr{Q}_{ext} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \tag{14}$$

is an element of $D(\mathscr{A}_{ext})$, and

$$\begin{bmatrix} (s\mathscr{E} - \mathscr{A}\mathscr{Q}) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \\ s\mathscr{E}_3\tilde{x}_3 \end{bmatrix} = (s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where $y_3 = \mathscr{A}_{3,ext}\left[\begin{smallmatrix} x_1 \\ x_2 \\ 0 \end{smallmatrix}\right]$. In particular,

$$(s\mathscr{E} - \mathscr{A}\mathscr{Q}) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and so $(s\mathscr{E} - \mathscr{A}\mathscr{Q})$ is surjective. Combined with its injectivity and equations (13)–(14), we see that $(s\mathscr{E} - \mathscr{A}\mathscr{Q})$ is boundedly invertible. □

From Theorem 8 we see that, if Assumption 4 holds for the adHDAE (7), then for the *reduced adHDAE*

$$\mathscr{E}\dot{x}(t) = \mathscr{A}\mathscr{Q}x(t) \tag{15}$$

with $\mathscr{A}$, $\mathscr{E}$, and $\mathscr{Q}$ defined in (10)–(12), the following conditions are satisfied.

Assumption 9

i) The state space is the Hilbert space $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$.

ii) $\mathscr{A} = \begin{bmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \end{bmatrix}$ *is dissipative on* $\mathbb{X}$.

iii) The operators $\mathscr{E}$ *and* $\mathscr{Q}$ *are of the form*

$$\mathscr{E} = \begin{bmatrix} \mathscr{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathscr{Q} = \begin{bmatrix} \mathscr{Q}_1 & 0 \\ 0 & \mathscr{Q}_2 \end{bmatrix}, \tag{16}$$

where $\mathscr{E}_1, \mathscr{Q}_1$ *and* $\mathscr{Q}_2$ *are bounded and boundedly invertible operators. Furthermore,* $\mathscr{E}_1^*\mathscr{Q}_1$ *is coercive, i.e., it is self-adjoint and* $\langle \mathscr{E}_1 x, \mathscr{Q}_1 x\rangle > \kappa\|x\|^2 > 0$ *for all nonzero* x.

iv) There exists an $s \in \mathbb{C}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ *such that* $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ *is boundedly invertible.*

Theorem 8 shows that in an adHDAE system (1) that satisfies Assumption 4 there exists a reduced subsystem for which Assumption 9 holds. In our next result we analyze the relation between the two sets of Assumptions 4 and 9. We show in particular that we can always extend an adHDAE of the form (15) satisfying Assumption 9 to a system of the form (1) satisfying Assumption 4 without changing the Hamiltonian.

Theorem 10 *Consider an adHDAE of the form (15) satisfying Assumption 9. Let* $\mathscr{A}_{ext}$ *with* $D(\mathscr{A}_{ext}) \subset \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3$ *be a dissipative extension of* $\mathscr{A}$ *such that (10) and (11) hold. Let* $\mathscr{E}_3$ *be a bounded and boundedly invertible operator on* $\mathbb{X}_3$*, and define* $\mathscr{E}_{ext}$ *and* $\mathscr{Q}_{ext}$ *as in (8). Then the triple* $(\mathscr{E}_{ext}, \mathscr{A}_{ext}, \mathscr{Q}_{ext})$ *satisfies Assumption 4 with the same Hamiltonian (9).*

Proof. It is clear that the Hamiltonian does not change, so it remains to show that $s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}$ is boundedly invertible. The equation

$$(s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is equivalent to the two equations

$$(s\mathscr{E} - \mathscr{A}\mathscr{Q}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ and } s\mathscr{E}_3 x_3 - \mathscr{A}_{3,ext} \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \\ 0 \end{bmatrix} = y_3.$$

Since the pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular we can determine x_1 and x_2 uniquely, and since $\mathscr{E}_3$ is boundedly invertible, x_3 is also uniquely determined when x_1, x_2 are fixed. Since these inverse mappings are bounded, we conclude that $s\mathscr{E}_{ext} - \mathscr{A}_{ext}\mathscr{Q}_{ext}$ is boundedly invertible. □

Based on Theorems 8 and 10 we see that we can reduce or extend regular adHDAEs when the Hamiltonian is not changed. For this reason from now on we only consider abstract DAEs without a component x_3, i.e., we study the adHDAE (15) under the Assumption 9, see [3, 30, 44] for the finite dimensional case. Note however, that for discretization methods and practical applications it is essential to keep the equation for x_3 for initial value consistency checks and to avoid that the solution for the variables x_1, x_2 drifts off from the solution manifold, see [25].

2.1 Regularity and singularity of adHDAEs

In this section we consider the regularity and singularity of the pair of operators

$$(\mathscr{E}, \mathscr{A}\mathscr{Q}) \tag{17}$$

associated with the adHDAE (15). We study the regularity of (17) under the first three conditions of Assumption 9. Using the fact that

$$s\mathscr{E} - \mathscr{A}\mathscr{Q} = \left(s \begin{bmatrix} \mathscr{E}_1\mathscr{Q}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} - \mathscr{A} \right) \begin{bmatrix} \mathscr{Q}_1 & 0 \\ 0 & \mathscr{Q}_2 \end{bmatrix} \tag{18}$$

and that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are bounded and boundedly invertible, the following lemma is immediate.

Lemma 11 *The operator* $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ *is boundedly invertible if and only if* $s\hat{\mathscr{E}} - \mathscr{A}$ *is boundedly invertible, where* $\hat{\mathscr{E}} = \begin{bmatrix} \mathscr{E}_1\mathscr{Q}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Furthermore, $\mathscr{E}_1^*\mathscr{Q}_1$ *is coercive if and only if* $\mathscr{E}_1\mathscr{Q}_1^{-1}$ *is coercive if and only if* $\mathscr{Q}_1\mathscr{E}_1^{-1}$ *is coercive.*

From this lemma we see that if we want to check the regularity of $(\mathscr{E}, \mathscr{A}\mathscr{Q})$, we may without loss of generality assume that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are the identity operators, and that $\mathscr{E}_1$ is coercive. We begin by showing that regularity implies that $\mathscr{A}$ is *maximally dissipative*, i.e., it is dissipative and for all $s > 0$ the operator $sI - \mathscr{A}$ is surjective.

Lemma 12 *Consider an abstract adHDAE of the form (1) satisfying Assumption 9. Then the operator $\mathscr{A}$ is maximally dissipative.*

Proof. If $s \in \mathbb{C}^+$ and since $\mathscr{E}_1\mathscr{Q}_1^{-1}$ is coercive, we have (see Lemma 11) that $\mathscr{A} - s\hat{\mathscr{E}}$ is dissipative. Since by assumption $\mathscr{A} - s\hat{\mathscr{E}}$ is boundedly invertible, by Lemmas 42 and 40 in the appendix we have that it is maximally dissipative. Since $s\hat{\mathscr{E}}$ is bounded this means that $\mathscr{A}$ is maximally dissipative. □

Theorem 13 *Consider a triple of operators $(\mathscr{E}, \mathscr{A}, \mathscr{Q})$, where we assume that these operators satisfy the first three conditions of Assumption 9. Then the following are equivalent.*

i) The pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular.
ii) For all $s \in \mathbb{C}^+$ the operator $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is boundedly invertible.
iii) There exists an $s \in \mathbb{C}^+$ such that the operator $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is boundedly invertible.
iv) The operator $\mathscr{A}$ is maximally dissipative, and there exists an $m_1 > 0$ such that

$$\left\| \begin{bmatrix} \mathscr{E} \\ \mathscr{A}\mathscr{Q} \end{bmatrix} x \right\| \geq m_1 \|x\| \text{ for all } \mathscr{Q}x \in D(\mathscr{A}). \tag{19}$$

v) The operator $\mathscr{A}$ is maximally dissipative, and there exists an $m_2 > 0$ such that

$$\left\| \begin{bmatrix} \mathscr{E}\mathscr{Q}^{-1} \\ \mathscr{A} \end{bmatrix} x \right\| \geq m_2 \|x\| \text{ for all } x \in D(\mathscr{A}). \tag{20}$$

Proof. It is clear that ii) implies iii), and iii) implies that $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular, and thus iii) implies i). So we start by proving that i) implies iv). By the given assumptions and since i) holds, Assumption 9 holds. Thus Lemma 12 gives that $\mathscr{A}$ is maximally dissipative. In particular it is densely defined and closed.

Let $s \in \mathbb{C}$ be such that $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is boundedly invertible, then for $x \in D(\mathscr{A}\mathscr{Q})$

$$\begin{aligned} \|x\| &= \|(s\mathscr{E} - \mathscr{A}\mathscr{Q})^{-1}(s\mathscr{E} - \mathscr{A}\mathscr{Q})x\| \leq M\|(s\mathscr{E} - \mathscr{A}\mathscr{Q})x\| \\ &= M\left\| \begin{bmatrix} sI & -I \end{bmatrix} \begin{bmatrix} \mathscr{E} \\ \mathscr{A}\mathscr{Q} \end{bmatrix} x \right\| \leq MM_1 \left\| \begin{bmatrix} \mathscr{E} \\ \mathscr{A}\mathscr{Q} \end{bmatrix} x \right\|. \end{aligned}$$

Since both $M = \|(s\mathscr{E} - \mathscr{A}\mathscr{Q})^{-1}\|$ and $M_1 = \left\| \begin{bmatrix} sI & -I \end{bmatrix} \right\|$ are nonzero, (19) follows. It remains to show that $\ker(\mathscr{E}) \cap \ker(\mathscr{A}^*\mathscr{Q}) = \{0\}$. Let $x \in \ker(\mathscr{E}) \cap \ker(\mathscr{A}^*\mathscr{Q})$, then $(\bar{s}\mathscr{Q}^*\mathscr{E} - \mathscr{Q}^*\mathscr{A}^*\mathscr{Q})x = 0$. Since $\mathscr{Q}^*\mathscr{E}$ is self-adjoint, this is the same as $(\bar{s}\mathscr{E}^*\mathscr{Q} - \mathscr{Q}^*\mathscr{A}^*\mathscr{Q})x = 0$, and so

$$\langle \mathscr{Q}x, (s\mathscr{E} - \mathscr{A}\mathscr{Q})y \rangle = 0 \text{ for all } \mathscr{Q}y \in D(\mathscr{A}).$$

Since $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is boundedly invertible its range equals $\mathbb{X}$ and thus $\mathscr{Q}x = 0$, and since $\mathscr{Q}$ is boundedly invertible, $x = 0$.

Since $\mathscr{Q}$ is boundedly invertible it is easy to see that items iv) and v) are equivalent. So it remains to show that iv) implies ii). To prove this, suppose that (19) holds, but that $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is not boundedly invertible for some $s \in \mathbb{C}$ with positive real part. Then we have the following possibilities:

(a) The operator $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is not injective.
(b) The range of $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is not dense in $\mathbb{X}$.
(c) The range of $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is dense but not equal to $\mathbb{X}$.

We will show that neither of these options is valid.

Case (a): If there exists $0 \neq x \in D(\mathscr{A}\mathscr{Q})$ such that $(s\mathscr{E} - \mathscr{A}\mathscr{Q})x = 0$ then consider $\langle (s\mathscr{E} - \mathscr{A}\mathscr{Q})x, \mathscr{Q}x\rangle$ which is zero, and thus

$$0 = s\langle \mathscr{E}x, \mathscr{Q}x\rangle - \langle \mathscr{A}\mathscr{Q}x, \mathscr{Q}x\rangle = s\langle \mathscr{E}\mathscr{Q}^{-1}\mathscr{Q}x, \mathscr{Q}x\rangle - \langle \mathscr{A}\mathscr{Q}x, \mathscr{Q}x\rangle.$$

Since $\mathrm{Re}(s) > 0$, $\mathscr{E}\mathscr{Q}^{-1}$ is non-negative (see Lemma 11) and since $\mathscr{A}$ is dissipative, taking the real part gives that

$$0 = \langle \mathscr{E}\mathscr{Q}^{-1}\mathscr{Q}x, \mathscr{Q}x\rangle.$$

Since $\mathscr{E}\mathscr{Q}^{-1}$ is a non-negative bounded operator, this gives that $\mathscr{Q}x \in \ker(\mathscr{E}\mathscr{Q}^{-1})$, or equivalently $x \in \ker(\mathscr{E})$.

Applying this in equation $(s\mathscr{E} - \mathscr{A}\mathscr{Q})x = 0$ gives $\mathscr{A}\mathscr{Q}x = 0$. So we have shown that $x \neq 0$ lies in $\ker(\mathscr{E}) \cap \ker(\mathscr{A}\mathscr{Q})$, which is a contradiction to (19).

Case (b): If there exists $0 \neq x \in \mathbb{X}$ such that

$$\langle x, (s\mathscr{E} - \mathscr{A}\mathscr{Q})y\rangle = 0 \text{ for all } \mathscr{Q}y \in D(\mathscr{A}), \tag{21}$$

then x lies in the domain of the dual operator, i.e., in $D((s\mathscr{E} - \mathscr{A}\mathscr{Q})^*)$, which equals $D(\mathscr{A}^*)$. Furthermore, since $D(\mathscr{A})$ is dense in $\mathbb{X}$, (21) implies that $0 = (s\mathscr{E} - \mathscr{A}\mathscr{Q})^* x = (\bar{s}\mathscr{E}^* - \mathscr{Q}^*\mathscr{A}^*)x$. Writing $x = \mathscr{Q}z$ and using the fact that $\mathscr{A}$ is maximally dissipative, and thus $\mathscr{A}^*$ is dissipative, we can proceed as in case (a) to obtain that $\mathscr{E}^*\mathscr{Q}z = 0$ and $\mathscr{Q}^*\mathscr{A}^*\mathscr{Q}z = 0$, or equivalently $\mathscr{E}^*x = 0$ and $\mathscr{Q}^*\mathscr{A}^*x = 0$. Since $\mathscr{Q}^*$ is boundedly invertible, this gives $\mathscr{A}^*x = 0$. The latter gives that

$$\langle x, \mathscr{A}y\rangle = 0 \text{ for all } y \in D(\mathscr{A}). \tag{22}$$

Since $\mathscr{A}$ is maximally dissipative, we know that there exists $y_x \in D(\mathscr{A})$ such that

$$(I - \mathscr{A})y_x = x. \tag{23}$$

Substituting this in (22) with $y = y_x$ gives

$$0 = \langle x, \mathscr{A}y_x\rangle = \langle (I - \mathscr{A})y_x, \mathscr{A}y_x\rangle = \langle y_x, \mathscr{A}y_x\rangle - \langle \mathscr{A}y_x, \mathscr{A}y_x\rangle.$$

The last inner product is obviously real and non-positive. The real part of the first term is also non-positive, and thus both terms must be zero. This gives in particular that $\mathscr{A}y_x = 0$, and by (23) that $y_x = x$. Note that we still have that $\mathscr{E}^*x = 0$.

Since $\mathscr{Q}$ is boundedly invertible, we can define $\tilde{x} = \mathscr{Q}^{-1}x$, and so $\tilde{x} \in \ker(\mathscr{A}\mathscr{Q}) \cap \ker(\mathscr{E}^*\mathscr{Q})$. Since $\mathscr{E}^*\mathscr{Q}$ is self-adjoint, this implies that $\tilde{x} \in \ker(\mathscr{Q}^*\mathscr{E})$, but since $\mathscr{Q}$ is boundedly invertible $\tilde{x} \in \ker(\mathscr{E})$. So substituting $\tilde{x}$ in (19) gives that $\tilde{x} = 0$ and hence $x = 0$, which is in contradiction to our assumption $x \neq 0$.

Case (c): Let $s \in \mathbb{C}^+$ be given and let $\mathscr{Q}x_n$ be a sequence in $D(\mathscr{A})$ such that $(s\mathscr{E} - \mathscr{A}\mathscr{Q})x_n \to z$ as $n \to \infty$ with $z \in \mathscr{X}$, but not in the range of $s\mathscr{E} - \mathscr{A}\mathscr{Q}$. Then by defining $x_{n,m} = x_n - x_m$, we have

$$(s\mathscr{E} - \mathscr{A}\mathscr{Q})x_{n,m} \to 0 \text{ as } n,m \to \infty. \tag{24}$$

If $\|x_{n,m}\| \to 0$ as $n,m \to \infty$, then x_n would be a Cauchy sequence, and thus converge to some x. In that case $z = (s\mathscr{E} - \mathscr{A}\mathscr{Q})x$, and thus in the range of $(s\mathscr{E} - \mathscr{A}\mathscr{Q})x$. Hence in this case we have a contradiction. So we assume that $\|x_{n,m}\|$ stays bounded away from zero for some sequence of indices $\{n,m\}$. In the remainder of the proof we consider this sequence.

Taking the inner product of (24) with $\frac{\mathscr{Q}x_{n,m}}{\|x_{n,m}\|}$, gives

$$0 = \lim_{n,m\to\infty} \left[s\langle \mathscr{E}x_{n,m}, \frac{\mathscr{Q}x_{n,m}}{\|x_{n,m}\|}\rangle - \langle \mathscr{A}\mathscr{Q}x_{n,m}, \frac{\mathscr{Q}x_{n,m}}{\|x_{n,m}\|}\rangle \right].$$

Since $\mathrm{Re}(s) > 0$ and $\mathscr{Q}^*\mathscr{E}$ is self-adjoint, taking the real part gives

$$0 = \lim_{n,m\to\infty} \left[\mathrm{Re}(s)\langle \mathscr{Q}^*\mathscr{E}x_{n,m}, \frac{x_{n,m}}{\|x_{n,m}\|}\rangle - \mathrm{Re}\left(\langle \mathscr{A}\mathscr{Q}x_{n,m}, \frac{\mathscr{Q}x_{n,m}}{\|x_{n,m}\|}\rangle \right) \right].$$

Both terms are nonnegative and since $\mathscr{Q}^*\mathscr{E} \geq 0$ we find that

$$\lim_{n,m\to\infty} \mathscr{Q}^*\mathscr{E}\frac{x_{n,m}}{\sqrt{\|x_{n,m}\|}} = 0 \Rightarrow \lim_{n,m\to\infty} \mathscr{E}\frac{x_{n,m}}{\sqrt{\|x_{n,m}\|}} = 0, \tag{25}$$

where we have used that $\mathscr{Q}$ is boundedly invertible. Applying this in (24) and using that $\|x_{n,m}\|$ stays bounded away from zero, we find that

$$\lim_{n,m\to\infty} \mathscr{A}\mathscr{Q}\frac{x_{n,m}}{\sqrt{\|x_{n,m}\|}} = 0. \tag{26}$$

Define $z_{n,m} = \frac{x_{n,m}}{\sqrt{\|x_{n,m}\|}}$, then by (24) and (25), equation (19) implies that $z_{n,m} \to 0$. However, by (19) this gives that

$$\lim_{n,m\to\infty} \|z_{n,m}\| = 0$$

which is equivalent to $\sqrt{\|x_{n,m}\|} \to 0$, which is a contradiction.

So we see that neither of the cases (a), (b), or (c) is possible, and hence item ii) holds. □

From Theorem 13 we can derive some easy consequences, but we begin by showing that the conditions as stated in item iv) and v) can be simplified when $\mathscr{A}$ has more structure.

Lemma 14 *Consider a triple of operators $(\mathscr{E}, \mathscr{A}, \mathscr{Q})$ that satisfy the first three conditions of Assumption 9. Assume further that $\mathscr{A}$ can be written as $\mathscr{A} = \mathscr{J} - \mathscr{R}$, with $\mathscr{J}$ skew-adjoint, i.e., $\mathscr{J}^* = -\mathscr{J}$ and $\mathscr{R}$ is bounded, self-adjoint and non-negative, then item v) in Theorem 13 is equivalent to*

v') There exists an $m_2 > 0$ such that

$$\left\| \begin{bmatrix} \mathscr{E}\mathscr{Q}^{-1} \\ \mathscr{J} \\ \mathscr{R} \end{bmatrix} x \right\| \geq m_2\|x\| \text{ for all } x \in D(\mathscr{J}). \tag{27}$$

Proof. Assume that v) holds, then we only have to show that (20) implies (27). If (27) would not hold, then there exits a sequence $\{x_n\}, n \in \mathbb{N}$ such that, $x_n \in D(\mathscr{J})$, $\|x_n\| = 1$ and $\mathscr{E}\mathscr{Q}^{-1}x_n$, $\mathscr{J}x_n$ and $\mathscr{R}x_n$ all converge to zero. This implies that $\mathscr{E}\mathscr{Q}^{-1}x_n$ and $\mathscr{A}x_n = (\mathscr{J} - \mathscr{R})x_n$ converge to zero, which is a contradiction.

Next we assume that v') holds. Then, since $\mathscr{A} = \mathscr{J} - \mathscr{R}$, with $\mathscr{R}$ is bounded and non-negative, and $\mathscr{J}$ skew-adjoint, we have that $D(\mathscr{A}) = D(\mathscr{A}^*)$ and both $\mathscr{A}$ and $\mathscr{A}^* = -\mathscr{J} - \mathscr{R}$ are dissipative, which implies that $\mathscr{A}$ is maximally dissipative.

Let $x_n \in D(\mathscr{J})$ be of norm one, and assume that $\mathscr{A}x_n \to 0$ as $n \to \infty$, then

$$\langle x_n, (\mathscr{J} - \mathscr{R})x_n \rangle \to 0$$

Taking the real part of this expression gives that $\langle x_n, \mathscr{R}x_n \rangle \to 0$. Since $\mathscr{R}$ is non-negative and bounded, this implies that $\mathscr{R}x_n \to 0$, see [7, Lemma A.3.88.c]. Combining this with $\mathscr{A}x_n \to 0$ gives that $\mathscr{J}x_n \to 0$. Hence from this we conclude that (27) implies (20). □

Note that similarly, the condition iv) in Theorem 13 can be replaced. Note further that for matrices or bounded operators a dissipative $\mathscr{A}$ can always be written as $\mathscr{A} = \mathscr{J} - \mathscr{R}$, with $\mathscr{J}$ skew-adjoint and $\mathscr{R}$ non-negative.

The result of Lemma 14 also holds when $\mathscr{A} = \mathscr{J} - \mathscr{R}$, with $\mathscr{J}$ skew-adjoint and bounded and $\mathscr{R}$ self-adjoint and non-negative. In that case we have $D(\mathscr{A}) = D(\mathscr{R})$.

Given the special form of $\mathscr{E}$ and $\mathscr{Q}$ the following is an easy consequence of Theorem 13.

Corollary 15 *Consider an adHDAE of the form* (15) *satisfying the conditions i)-iii) in Assumption 9 and define*

$$\mathscr{E}_I := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{28}$$

Then the following are equivalent.

i) There exists an $s \in \mathbb{C}^+$ such that the operator $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is boundedly invertible.
ii) There exists an $s \in \mathbb{C}^+$ such that the operator $s\mathscr{E}_I - \mathscr{A}$ is boundedly invertible.

Proof. From Theorem 13 we have to show that we may replace $\mathscr{E}\mathscr{Q}^{-1}$ by $\mathscr{E}_I$. This follows since

$$\mathscr{E}_I = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathscr{Q}_1\mathscr{E}_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{E}_1\mathscr{Q}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

where we have used the invertiblity of $\mathscr{E}_1$ and $\mathscr{Q}$. So $\mathscr{E}\mathscr{Q}^{-1}$ and $\mathscr{E}_I$ are boundedly invertible related to each other, and this implies that in Theorem 13 part iv) and v) we may do the replacements. □

We have shown that the regularity of the pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is equivalent to that of $(\mathscr{E}_I, \mathscr{A})$. However, this may still be a difficult condition to check. In the following lemma we derive conditions under which this follows from the maximal dissipativity of $\mathscr{A}$.

Lemma 16 *Consider an adHDAE of the form* (15) *satisfying the conditions i)-iii) in Assumption 9. If there exists an $\varepsilon > 0$ such that $\mathscr{A} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is maximally dissipative, then $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular.*

Proof. Since $\mathscr{A} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is maximally dissipative, we know that $\mathscr{A} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \delta \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ is boundedly invertible for every $\delta > 0$. Choosing $\delta = \varepsilon$, we see that this implies that $\varepsilon\mathscr{E}_I - \mathscr{A}$ is boundedly invertible. By Corollary 15 it follows that this is equivalent to $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ being regular. □

We end this section with a few observations and additional results.

In the finite-dimensional case it has been shown in [28] that if $\mathscr{Q}$ is injective and the pair is singular then the three matrices $\mathscr{E}, \mathscr{J}\mathscr{Q}, \mathscr{R}\mathscr{Q}$ have a common nullspace. Here $\mathscr{J} = \frac{1}{2}(\mathscr{A} - \mathscr{A}^*)$ and $\mathscr{R} = -\frac{1}{2}(\mathscr{A} + \mathscr{A}^*)$. However, this is not true if $\mathscr{Q}$ is not injective.

Example 17 *Consider the matrices*

$$\mathscr{E} = \begin{bmatrix} e_{11} & e_{12} \\ 0 & 0 \end{bmatrix}, \quad \mathscr{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{Q} = \begin{bmatrix} 0 & 0 \\ q_{21} & q_{22} \end{bmatrix}.$$

Then

$$\mathscr{J}\mathscr{Q} = \begin{bmatrix} -q_{21} & -q_{22} \\ 0 & 0 \end{bmatrix}, \text{ and } s\mathscr{E} - \mathscr{J}\mathscr{Q} = \begin{bmatrix} se_{11} + q_{21} & se_{12} + q_{22} \\ 0 & 0 \end{bmatrix},$$

and so the pair $(\mathscr{E}, \mathscr{J}\mathscr{Q})$ is singular. Furthermore,

$$\mathscr{E}^*\mathscr{Q} = \begin{bmatrix} e_{11} & 0 \\ e_{12} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is symmetric, and positive semidefinite. However, $\mathscr{E}$ and $\mathscr{J}\mathscr{Q}$ do not have a common kernel.

Since our aim was to study regularity, i.e., boundedly invertibility of $s\mathscr{E} - \mathscr{A}\mathscr{Q}$, we had to check conditions for injectivity and surjectivity. However, separate conditions can also be obtained.

Lemma 18 *Consider an adHDAE of the form (1) satisfying the conditions i)-iii) in Assumption 9, and let $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{Q}}$ satisfy the same assumptions as $\mathscr{E}$ and $\mathscr{Q}$ in Assumption 9, respectively. Furthermore, let $s, \tilde{s} \in \mathbb{C}^+$. Then the following assertions hold.*

a) $(\tilde{s}\tilde{\mathscr{E}} - \mathscr{A}\tilde{\mathscr{Q}})$ is injective if and only if $(s\mathscr{E} - \mathscr{A}\mathscr{Q})$ is. Furthermore, this holds if and only if $\mathscr{A}\left[\begin{smallmatrix}0\\x_2\end{smallmatrix}\right] = 0$ implies $x_2 = 0$.

b) Let $\mathscr{A}$ be maximally dissipative. The range of $\tilde{s}\tilde{\mathscr{E}} - \mathscr{A}\tilde{\mathscr{Q}}$ is dense if and only if the range of $s\mathscr{E} - \mathscr{A}\mathscr{Q}$ is dense. Furthermore, this holds if and only if $\mathscr{A}^*\left[\begin{smallmatrix}0\\z_2\end{smallmatrix}\right] = 0$ implies $z_2 = 0$.

Proof. The proofs are similar to the corresponding parts of the proof of Theorem 13. □

2.2 Special block operators

As a prototypical example of adHDAEs, in this section we study special block operators pairs, as they arise e.g. in Stokes and Oseen equations that have been formulated in [12], or [36], as abstract DAE. Similar abstract block DAE operators arise also in the study of the Euler equations in gas transport [10, 11].

In the following $\mathscr{L}(\mathbb{W}, \mathbb{Y})$ denotes the space of bounded, linear operators between Hilbert spaces $\mathbb{W}$ and $\mathbb{Y}$. Furthermore, $\mathscr{L}(\mathbb{W}) = \mathscr{L}(\mathbb{W}, \mathbb{W})$.

Let $\mathbb{V}$ be a real Hilbert space such that $\mathbb{V} \hookrightarrow \mathbb{X}_1 = \mathbb{X}_1^* \hookrightarrow \mathbb{V}^*$, i.e., they form a *Gelfand triple*, see e.g., [47]. Let $\mathscr{A}_0 \in \mathscr{L}(\mathbb{V}, \mathbb{V}^*)$, $\mathscr{B}_0 \in \mathscr{L}(\mathbb{U}, \mathbb{V}^*)$, where $\mathbb{U}$ is a second (real) Hilbert space. So $\mathscr{B}_0^* \in \mathscr{L}(\mathbb{V}, \mathbb{U})$, where we have identified $\mathbb{U}^*$ with $\mathbb{U}$. Finally, with these operators and $\mathscr{D}_0 \in \mathscr{L}(\mathbb{U})$, we define the block operator

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 & \mathscr{B}_0 \\ -\mathscr{B}_0^* & -\mathscr{D}_0 \end{bmatrix} \tag{29}$$

with domain

$$D(\mathscr{A}) = \{\left[\begin{smallmatrix}v\\u\end{smallmatrix}\right] \in \mathbb{V} \oplus \mathbb{U} \mid \mathscr{A}_0 v + \mathscr{B}_0 u \in \mathbb{X}_1\}. \tag{30}$$

For this operator $\mathscr{A}$, we study the pair $(\mathscr{E}_I, \mathscr{A})$ with $\mathscr{E}_I$ as in (28) and begin our analysis with two simple lemmas.

Lemma 19 *Consider the operator $\mathscr{A}$ as in (29) with its domain as in (30). Assume that $-\mathscr{D}_0$ and $\mathscr{A}_0$ are dissipative, i.e.,*

$$\langle \mathscr{A}_0 v, v\rangle_{\mathbb{V}^*,\mathbb{V}} \le 0 \text{ for all } v \in \mathbb{V}, \ \langle -D_0 u, u\rangle_{\mathbb{U}^*,\mathbb{U}} \le 0 \text{ for all } u \in \mathbb{U}, \tag{31}$$

then $\mathscr{A}$ is dissipative on $\mathbb{X}_1 \oplus \mathbb{U}$.

Proof. To show that $\mathscr{A}$ is dissipative on $\mathbb{X}_1 \oplus \mathbb{U}$, we choose $\left[\begin{smallmatrix}v\\u\end{smallmatrix}\right] \in D(\mathscr{A})$. Then we have

$$\begin{aligned}
\langle \mathscr{A}\begin{bmatrix} v \\ u\end{bmatrix}, \begin{bmatrix} v \\ u\end{bmatrix}\rangle_{\mathbb{X}_1\oplus\mathbb{U}} + \langle \begin{bmatrix} v \\ u\end{bmatrix}, \mathscr{A}\begin{bmatrix} v \\ u\end{bmatrix}\rangle_{\mathbb{X}_1\oplus\mathbb{U}}& \\
= \langle \mathscr{A}_0 v + \mathscr{B}_0 u, v\rangle_{\mathbb{X}_1} + \langle v, \mathscr{A}_0 v + \mathscr{B}_0 u\rangle_{\mathbb{X}_1} +& \\
\langle -\mathscr{B}_0^* v - \mathscr{D}_0 u, u\rangle_{\mathbb{U}} + \langle u, -\mathscr{B}_0^* v - \mathscr{D}_0 u\rangle_{\mathbb{U}}& \\
= \langle \mathscr{A}_0 v + \mathscr{B}_0 u, v\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle v, \mathscr{A}_0 v + \mathscr{B}_0 u\rangle_{\mathbb{V},\mathbb{V}^*} -& \\
\langle \mathscr{B}_0^* v, u\rangle_{\mathbb{U}} - \langle u, \mathscr{B}_0^* v\rangle_{\mathbb{U}} - \langle \mathscr{D}_0 u, u\rangle_{\mathbb{U}} - \langle u, \mathscr{D}_0 u\rangle_{\mathbb{U}}& \\
= \langle \mathscr{A}_0 v, v\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle \mathscr{B}_0 u, v\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle v, \mathscr{A}_0 v\rangle_{\mathbb{V},\mathbb{V}^*} + \langle v, \mathscr{B}_0 u\rangle_{\mathbb{V},\mathbb{V}^*}& \\
- \langle v, \mathscr{B}_0 u\rangle_{\mathbb{V},\mathbb{V}^*} - \langle \mathscr{B}_0 u, v\rangle_{\mathbb{V}^*,\mathbb{V}} - \langle \mathscr{D}_0 u, u\rangle_{\mathbb{U}} - \langle u, \mathscr{D}_0 u\rangle_{\mathbb{U}}& \\
= \langle \mathscr{A}_0 v, v\rangle_{\mathbb{V}^*,\mathbb{V}} + \langle v, \mathscr{A}_0 v\rangle_{\mathbb{V},\mathbb{V}^*} - \langle \mathscr{D}_0 u, u\rangle_{\mathbb{U}} - \langle u, \mathscr{D}_0 u\rangle_{\mathbb{U}} \le 0,&
\end{aligned}$$

where we have used (31). Thus we have proved the assertion. □

Therefore, if we choose $\mathscr{E} = \mathscr{E}_I$, $\mathscr{Q} = I$, and $\mathscr{A}$ as in (29)–(30) to be dissipative, then the conditions i)–iii) of Assumption 9 are satisfied. In this setting, we study the injectivity of $(\mathscr{E}_I, \mathscr{A})$.

Lemma 20 *Consider the operator $\mathscr{A}$ with its domain as in (29) and (30). Suppose that $\mathscr{A}$ is dissipative and one of the following two conditions holds:*

a) $\mathscr{B}_0$ is injective, or
b) the range of $\mathscr{B}_0$ intersected with $\mathbb{X}_1$ contains only the zero element,

then $\mathscr{E}_I - \mathscr{A}$ is injective.

Proof. We use Lemma 18 a) to prove the assertion and study the equation $\mathscr{A}\left[\begin{smallmatrix}0\\u\end{smallmatrix}\right] = 0$. Note that this implies in particular that $\left[\begin{smallmatrix}0\\u\end{smallmatrix}\right] \in D(\mathscr{A})$. By (30) this gives the condition that $\mathscr{A}_0 0 + \mathscr{B}_0 u = \mathscr{B}_0 u \in \mathbb{X}_1$. So if b) holds, this can only happen when $u = 0$. If $\mathscr{B}_0$ can map into $\mathbb{X}_1$, then the equation $\mathscr{A}\left[\begin{smallmatrix}0\\u\end{smallmatrix}\right] = 0$ implies $\mathscr{B}_0 u = 0$. Then a) gives $u = 0$, and the proof is complete. □

Note that condition b) in Lemma 20 is sometimes rephrased as $\mathscr{B}_0$ *is completely unbounded*.

To show that $\mathscr{E}_I - \mathscr{A}$ is boundedly invertible, we need stronger conditions on $\mathscr{B}_0$ and $\mathscr{A}_0$. We say that $\mathscr{A}_0$ *satisfies a Gårding inequality* with respect to $\mathbb{X}_1$ and $\mathbb{V}$, if there exists an $\alpha_1 > 0$ such for all $v \in \mathbb{V}$ the inequality

$$\|v\|_{\mathbb{X}_1}^2 + |\langle \mathscr{A}_0 v, v\rangle_{\mathbb{V}^*,\mathbb{V}}| \geq \alpha_1 \|v\|_{\mathbb{V}}^2 \tag{32}$$

holds. Note that since $\mathscr{A}_0 \in \mathscr{L}(\mathbb{V}, \mathbb{V}^*)$ and $\mathbb{V} \hookrightarrow \mathbb{X}_1$, we always have that

$$\|v\|_{\mathbb{X}_1}^2 + |\langle \mathscr{A}_0 v, v\rangle_{\mathbb{V}^*,\mathbb{V}}| \leq \alpha_2 \|v\|_{\mathbb{V}}^2$$

for some $\alpha_2 > 0$.

Lemma 21 *Let $\mathscr{A}_0 \in \mathscr{L}(\mathbb{V}, \mathbb{V}^*)$ be dissipative and satisfy the Gårding inequality (32). Then $i_{\mathbb{V}} - \mathscr{A}_0$ is a boundedly invertible operator from $\mathbb{V}$ to $\mathbb{V}^*$. Here $i_{\mathbb{V}}$ is the inclusion map from $\mathbb{V}$ into $\mathbb{V}^*$, i.e., $i_{\mathbb{V}}(v) = v$, for $v \in \mathbb{V}$.*

Proof. See e.g. [Section 6.5] in [19]. □

We will now present two theorems which give sufficient conditions for $(\mathscr{E}_I, \mathscr{A})$ to be regular.

Theorem 22 *Consider the operator $\mathscr{A}$ given by (29) and (30). Let $\mathscr{A}_0$ and $-\mathscr{D}_0$ be dissipative, and assume further that $\mathscr{A}_0$ satisfies the Gårding inequality (32). Finally, let $\left[\begin{smallmatrix}\mathscr{B}_0\\ \mathscr{D}_0\end{smallmatrix}\right]$ be injective and have closed range, i.e., there exists $\beta > 0$ such that for all $u \in \mathbb{U}$*

$$\left\| \begin{bmatrix} \mathscr{B}_0 \\ \mathscr{D}_0 \end{bmatrix} u \right\|_{\mathbb{V}^* \oplus \mathbb{U}} \geq \beta \|u\|_{\mathbb{U}}. \tag{33}$$

Under these conditions, $\mathscr{E}_I - \mathscr{A}$ is boundedly invertible.

Moreover, $\mathscr{E}_I - \mathscr{A}$ is boundedly invertible if and only if the Schur complement $\mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 + \mathscr{D}_0$ *is boundedly invertible.*

Proof. The proof consists of several parts. We begin by showing that the Schur complement function $G(1) := \mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 + \mathscr{D}_0$ is accretive, i.e., for all $u \in \mathbb{U}$ it holds that

$$\mathrm{Re}\langle G(1)u, u\rangle \geq 0. \tag{34}$$

We have

$$\begin{aligned}\langle G(1)u, u\rangle &= \langle \mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 u, u\rangle + \langle \mathscr{D}_0 u, u\rangle \\ &= \langle (i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 u, \mathscr{B}_0 u\rangle_{\mathbb{V},\mathbb{V}^*} + \langle \mathscr{D}_0 u, u\rangle \\ &= \langle v, (i_{\mathbb{V}} - \mathscr{A}_0)v\rangle_{\mathbb{V},\mathbb{V}^*} + \langle \mathscr{D}_0 u, u\rangle,\end{aligned}$$

with $v = (i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 u$. Since $-\mathscr{D}_0$ and $\mathscr{A}_0$ are dissipative, inequality (34) then follows.

Next we show that $\mathscr{A}$ is maximally dissipative. For this we look at the equation

$$(I - \mathscr{A})\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ y \end{bmatrix}$$

for an arbitrary $x_1 \in \mathbb{X}_1$ and $y \in \mathbb{U}$, where we search a solution $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \in D(\mathscr{A})$. The above equation can be written as two equations

$$(I - \mathscr{A}_0)v - \mathscr{B}_0 u = x_1 \quad \text{and } \mathscr{B}_0^* v + \mathscr{D}_0 u + u = y.$$

Since $\mathbb{X}_1 \subset \mathbb{V}^*$, we have by Lemma 21 that the first equation has the solution $v \in \mathbb{V}$ given by

$$v = (i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 u + (i_{\mathbb{V}} - \mathscr{A}_0)^{-1} x_1. \tag{35}$$

Substituting this in the second equation leads to the following equation for u

$$\mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1}\mathscr{B}_0 u + \mathscr{D}_0 u + u + \mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1} x_1 = y,$$

which we can write as

$$(G(1) + I)u = y - \mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A}_0)^{-1} x_1. \tag{36}$$

By (34) we have that $-G(1)$ is a dissipative operator which is bounded, and thus maximally dissipative. Hence for every $x_1 \in \mathbb{X}_1$ and $y \in \mathbb{U}$ the equation (36) has a unique solution, depending continuously on x_1 and y. Now v is given by (35) which depends continuously on u and x_1, and thus on y and x_1. It remains to show that $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \in D(\mathscr{A})$. This follows directly since $\mathscr{A}_0 v + \mathscr{B}_0 u = -x_1 + v$. So we conclude that $\mathscr{A}$ is maximally dissipative.

Next we show that $\mathscr{A}$ satisfies (19). We know that we only have to show this for $\mathscr{E}_I$, see Corollary 15. If (19) would not hold, then there exists a sequence $\left[\begin{smallmatrix} v_n \\ u_n \end{smallmatrix}\right] \in \mathbb{X}_1 \oplus \mathbb{U}$ of norm 1, such that $\left[\begin{smallmatrix} v_n \\ u_n \end{smallmatrix}\right] \in D(\mathscr{A})$ and $\mathscr{E}_I \left[\begin{smallmatrix} v_n \\ u_n \end{smallmatrix}\right]$, $\mathscr{A} \left[\begin{smallmatrix} v_n \\ u_n \end{smallmatrix}\right]$ both converge to zero. This can equivalently be formulated as $v_n \to 0$ in $\mathbb{X}_1$ and

$$\mathscr{A}_0 v_n + \mathscr{B}_0 u_n \to 0 \text{ in } \mathbb{X}_1, \quad \mathscr{B}_0^* v_n + \mathscr{D}_0 u_n \to 0 \text{ in } \mathbb{U}. \tag{37}$$

We have the following equalities

$$\begin{aligned}&\langle v_n, \mathscr{A}_0 v_n + \mathscr{B}_0 u_n\rangle_{\mathbb{X}_1} - \langle u_n, \mathscr{B}_0^* v_n + \mathscr{D}_0 u_n\rangle \\ &\quad = \langle v_n, \mathscr{A}_0 v_n\rangle_{\mathbb{V},\mathbb{V}^*} + \langle v_n, \mathscr{B}_0 u_n\rangle_{\mathbb{V},\mathbb{V}^*} - \langle u_n, \mathscr{B}_0^* v_n\rangle_{\mathbb{U}} - \langle u_n, \mathscr{D}_0 u_n\rangle \\ &\quad = \langle v_n, \mathscr{A}_0 v_n\rangle_{\mathbb{V},\mathbb{V}^*} - \langle u_n, \mathscr{D}_0 u_n\rangle_{\mathbb{U}}.\end{aligned} \tag{38}$$

By (37) both summands in the left-most term converge to zero, and thus also the sum in the right-most term. Since the spaces are real and the operators $\mathscr{A}_0$ and $-\mathscr{D}_0$ are dissipative, $-\langle u_n, \mathscr{D}_0 u_n\rangle_{\mathbb{U}}$ and $\langle v_n, \mathscr{A}_0 v_n\rangle_{\mathbb{V},\mathbb{V}^*}$ take values in $(-\infty, 0]$. This shows that $\langle v_n \mathscr{A}_0 v_n\rangle_{\mathbb{V},\mathbb{V}^*} \to 0$ as $n \to \infty$. Combining this with $v_n \to 0$ in $\mathbb{X}_1$ and the Gårding inequality (32) gives $v_n \to 0$ in $\mathbb{V}$. Since $\mathscr{A}_0$ is bounded from $\mathbb{V}$ to $\mathbb{V}^*$, and since $\mathbb{X}_1 \subset\!\!\to \mathbb{V}^*$, we find that $\mathscr{A}_0 v_n \to 0$ in $\mathbb{V}^*$ as $n \to \infty$. The first relation in equation (37) gives that $\mathscr{B}_0 u_n \to 0$ in $\mathbb{V}^*$.

Since $v_n \to 0$ in $\mathbb{V}$ and since $\mathscr{B}_0$ is bounded from $\mathbb{V}$ to $\mathbb{U}$, we have that $\mathscr{B}_0 v_n \to 0$ in $\mathbb{U}$. The second relation in equation (37) gives that $\mathscr{D}_0 u_n \to 0$ in $\mathbb{U}$. Since $\mathscr{B}_0 u_n$ and $\mathscr{D}_0 u_n$ converge to zero, inequality (33) gives that $u_n \to 0$. Combined with $v_n \to 0$ in $\mathbb{X}_1$ this is in contradiction to the assumption that $\|\left[\begin{smallmatrix} v_n \\ u_n \end{smallmatrix}\right]\|_{\mathbb{X}_1 \oplus \mathbb{U}} = 1$. Hence (19) holds.

So we have shown that $(\mathscr{E}_I, \mathscr{A})$ satisfies the condition of Theorem 13.iv), and thus also that $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular. It remains to prove the last assertion of the theorem.

To prove that the invertibility of $G(1)$ implies the invertibility of $\mathscr{E}_I - \mathscr{A}$, we proceed similar to the first item in this proof. Namely, the equation $(\mathscr{E}_I - \mathscr{A})\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ y \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \in D(\mathscr{A})$ gives that v is given by (35) and u satisfies (see also (36))

$$-G(1)u = y + \mathscr{B}_0^*(i_{\mathbb{V}} - \mathscr{A})^{-1} x_1$$

From this and equation (35) it follows that $\mathscr{E}_I - \mathscr{A}$ is boundedly invertible when $G(1)$ is. It remains to show the opposite direction.

Assuming that $\mathscr{E}_I - \mathscr{A}$ is boundedly invertible gives that there is a unique and continuous mapping from $y \in \mathbb{U}$ to $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \in \mathbb{X}_1 \oplus \mathbb{U}$ such that $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \in D(\mathscr{A})$, and

$$(\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

Using Lemma 21 we can solve this equation, and find $v = (i_{\mathbb{V}} - \mathscr{A})^{-1}\mathscr{B}_0 u$ and $y = G(1)u$. This gives that there exists a continuous mapping from y to u, and thus $G(1)$ is boundedly invertible. □

From the above proof, we see that $\mathscr{D}_0$ being self-adjoint, which is a typical property in many applications, see e.g. [12], was only needed in one step. Alternatively, we could have assumed that $\mathscr{A}_0$ was self-adjoint. Of course both operators need to be dissipative. Property (34) is a special case of a general property which these systems likely have, namely that $G(s)$ is positive real, i.e., $\mathrm{Re}\langle G(s)u, u\rangle \geq 0$ whenever $s \in \mathbb{C}^+$.

Remark 23 *In a recent paper, [36], it is shown that the formulation that we have discussed can also be formulated in terms of system nodes, see Definition 4.7.2 in [41]. The assumption on $\mathscr{A}$ as stated in Lemma 16 is the condition used in [48].*

Using the property of the block structured pencil in (29) we have the following well-known inf-sup conditions when $\mathscr{D}_0 = 0$.

Theorem 24 *Consider the operator in (29) and define $\mathbb{V}_0 \subset \mathbb{V}$ as $\mathbb{V}_0 = \ker \mathscr{B}_0^*$. Then*

$$\inf_{0 \neq v \in \mathbb{V}_0} \sup_{0 \neq w \in \mathbb{V}_0} \frac{\langle \mathscr{A}_0 v, w\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|\|w\|} \geq \alpha > 0, \quad \inf_{0 \neq v \in \mathbb{V}_0} \sup_{0 \neq w \in \mathbb{V}_0} \frac{\langle \mathscr{A}_0 w, v\rangle_{\mathbb{V}^*,\mathbb{V}}}{\|v\|\|w\|} \geq \alpha > 0, \tag{39}$$

$$\inf_{0 \neq u \in U} \sup_{0 \neq v \in \mathbb{V}} \frac{\langle \mathscr{B}_0 u, v\rangle_{\mathbb{V}^*,V}}{\|u\|_{\mathbb{U}}\|v\|_{\mathbb{V}}} \geq \gamma > 0, \tag{40}$$

and the pair $(\mathscr{E}_I, \mathscr{A})$ is regular.

Proof. See e.g. [43]. □

3 Existence of solutions on the whole space

In this section we study the solution of adHDAEs of the form (15). Since x_2 is a constraint to x_1, we concentrate on the solution theory for x_1 first. Our first result is based on the extra assumption that the last row in (15) does not impose a condition on x_1, i.e., for every x_1 there exists an x_2 such that this condition is satisfied. This implies that the algebraic equations impose no restriction on the state space $\mathbb{X}_1$. The case in which that may happen is studied in Section 5.

We define the following reduced state space

$$\mathbb{X}_{1,\mathscr{E}\mathscr{Q}} = \mathbb{X}_1 \text{ with inner product } \langle x_1, \tilde{x}_1\rangle_{\mathscr{E}\mathscr{Q}} = \langle x_1, \mathscr{E}_1^*\mathscr{Q}_1\tilde{x}_1\rangle, \tag{41}$$

where the second inner product is the standard inner product of $\mathbb{X}_1$. Since $\mathscr{E}_1^*\mathscr{Q}_1$ is coercive, the new norm is equivalent to the original one.

Theorem 25 *Consider a adHDAE system of the form (15) with $(\mathscr{E},\mathscr{A}\mathscr{Q})$ regular satisfying Assumption 9 and assume that whenever $\begin{bmatrix} 0 \\ x_2\end{bmatrix} \in D(\mathscr{A})$ is such that $\mathscr{A}_2\begin{bmatrix} 0 \\ x_2\end{bmatrix} = 0$, then $x_2 = 0$.*

Under these assumptions, the operator $\mathscr{A}_{red} : D(\mathscr{A}_{red}) \subset \mathbb{X}_{1,\mathscr{E}\mathscr{Q}} \to \mathbb{X}_{1,\mathscr{E}\mathscr{Q}}$ generates a contraction semigroup on the reduced space $\mathbb{X}_{1,\mathscr{E}\mathscr{Q}}$, where the domain $D(\mathscr{A}_{red})$ is defined as

$$D(\mathscr{A}_{red}) = \{x_1 \in \mathbb{X}_{1,\mathscr{E}\mathscr{Q}} \mid \exists\, x_2 \in \mathscr{X}_2 \text{ such that } \begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix} \in D(\mathscr{A}) \text{ and } \mathscr{A}_2\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix} = 0\}, \tag{42}$$

and for $x_1 \in D(\mathscr{A}_{red})$ the action of $\mathscr{A}_{red}$ is defined as

$$\mathscr{A}_{red}x_1 = \mathscr{E}_1^{-1}\mathscr{A}_1\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}. \tag{43}$$

Proof. First we have to prove that $\mathscr{A}_{red}$ is well-defined. So if for a given $x_1 \in D(\mathscr{A}_{red})$ we have that x_2 and $\tilde{x}_2$ are such that the condition of the domain are satisfied for $\begin{bmatrix} x_1 \\ x_2\end{bmatrix}$ and $\begin{bmatrix} x_1 \\ \tilde{x}_2\end{bmatrix}$, then by the linearity of $\mathscr{A}_2$, we have that

$$\mathscr{A}_2\begin{bmatrix} 0 \\ \mathscr{Q}_2(x_2-\tilde{x}_2)\end{bmatrix} = 0.$$

By assumption, this implies that $\mathscr{Q}_2(x_2-\tilde{x}_2) = 0$, and since $\mathscr{Q}_2$ is invertible, then $x_2-\tilde{x}_2 = 0$. So there exists at most one x_2 to every $x_1 \in D(\mathscr{A}_{red})$, and hence $\mathscr{A}_{red}$ is well-defined.

Using that $\mathscr{E}_1^*\mathscr{Q}_1 = (\mathscr{E}_1^*\mathscr{Q}_1)^*$, we have

$$\begin{aligned}
\langle \mathscr{A}_{red}x_1, x_1\rangle_{\mathscr{E}\mathscr{Q}} + \langle x_1, \mathscr{A}_{red}x_1,\rangle_{\mathscr{E}\mathscr{Q}} =& \langle \mathscr{A}_{red}x_1, \mathscr{E}_1^*\mathscr{Q}_1x_1\rangle + \langle \mathscr{E}_1^*\mathscr{Q}_1x_1, \mathscr{A}_{red}x_1\rangle \\
=& \langle \mathscr{E}_1^{-1}\mathscr{A}_1\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}, \mathscr{E}_1^*\mathscr{Q}_1x_1\rangle + \langle \mathscr{E}_1^*\mathscr{Q}_1x_1, \mathscr{E}_1^{-1}\mathscr{A}_1\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}\rangle \\
=& \langle \mathscr{A}\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}, \begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}\rangle + \langle \begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}, \mathscr{A}\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}\rangle \\
\le& 0,
\end{aligned}$$

where we have used $\mathscr{A}_2\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix} = 0$ and the dissipativity of $\mathscr{A}$. Hence $\mathscr{A}_{red}$ is dissipative.

Now we show that $sI - \mathscr{A}_{red}$ is onto for $s \in \mathbb{C}^+$. Given $\begin{bmatrix} y_1 \\ 0\end{bmatrix} \in \mathbb{X}$. Then by assumption, see also Corollary 15, we know that there exists a $\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \in D(\mathscr{A}\mathscr{Q})$ such that

$$\begin{bmatrix} \mathscr{E}_1y_1 \\ 0\end{bmatrix} = (s\mathscr{E} - \mathscr{A}\mathscr{Q})\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = s\begin{bmatrix} \mathscr{E}_1x_1 \\ 0\end{bmatrix} - \mathscr{A}\begin{bmatrix} \mathscr{Q}_1x_1 \\ \mathscr{Q}_2x_2\end{bmatrix}. \tag{44}$$

The last row of this expression gives that

$$\mathscr{A}_2 \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{bmatrix} = 0$$

and so $x_1 \in D(\mathscr{A}_{red})$. The top row of (44) gives

$$s\mathscr{E}_1 x_1 - \mathscr{A}_1 \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{bmatrix} = \mathscr{E}_1 y_1$$

or equivalently (using (42) and that $\mathscr{E}_1$ is boundedly invertible) $(sI - \mathscr{A}_{red})x_1 = y_1$. This gives that $sI - \mathscr{A}_{red}$ is surjective for $s \in \mathbb{C}^+$. By the Lumer-Phillips Theorem, see e.g. [41], we conclude that $\mathscr{A}_{red}$ generates a contraction semigroup on $\mathbb{X}_1$. □

In the proof of Theorem 25, we did not use the regularity of the pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ to show that $\mathscr{A}_{red}$ is well-defined and dissipative. It was used only to prove the surjectivity of $sI - \mathscr{A}_{red}$. Since the latter is the property we want for $\mathscr{A}_{red}$, we can ask if our regularity assumption is not too strong. The following lemma shows that under a mild condition the two properties are equivalent.

Lemma 26 *Let the first three conditions of Assumption 9 hold. Furthermore, we assume that whenever* $\left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right] \in D(\mathscr{A})$ *is such that* $\mathscr{A}_2 \left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right] = 0$*, then* $x_2 = 0$*. Consider the operator* $\mathscr{A}_{red}$ *on the domain (42) with the action (43). Then the following are equivalent:*

a) The pair $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ *is regular.*
b) $\mathscr{A}$ *is closed,* $\mathscr{A}_2 : D(\mathscr{A}) \mapsto \mathbb{X}_2$ *is surjective, and there exists an* $s \in \mathbb{C}^+$ *such* $sI - \mathscr{A}_{red}$ *is boundedly invertible;*
c) $\mathscr{A}$ *is closed,* $\mathscr{A}_2 : D(\mathscr{A}) \mapsto \mathbb{X}_2$ *is surjective, and there exists an* $s \in \mathbb{C}^+$ *such* $sI - \mathscr{A}_{red}$ *is surjective;*
d) $\mathscr{A}$ *is closed,* $\mathscr{A}_2 : D(\mathscr{A}) \mapsto \mathbb{X}_2$ *is surjective and* $\mathscr{A}_{red}$ *is maximally dissipative.*

Proof. By Corollary 15 we only have to prove the equivalences for the pair $(\mathscr{E}_I, \mathscr{A})$. From the assumptions it follows that $\mathscr{A}_{red}$ is well-defined and dissipative, see the proof of Theorem 25.

$a) \Rightarrow b)$: By Lemma 12 $\mathscr{A}$ is maximally dissipative and so it is closed. The last part follows from Theorem 25, since if $\mathscr{A}_{red}$ generates a contraction semigroup, then $sI - \mathscr{A}_{red}$ is boundedly invertible for all $s \in \mathbb{C}^+$, see also Section 7. It remains to show that $\mathscr{A}_2$ is surjective. Since $(\mathscr{E}_I, \mathscr{A})$ is regular, $s\mathscr{E}_I - \mathscr{A}$ is surjective. This immediately implies that $\mathscr{A}_2$ is surjective.

$b) \Leftrightarrow d)$: This equivalence follows from the fact that a dissipative operator $\mathscr{A}_{red}$ is maximally dissipative if and only if $sI - \mathscr{A}_{red}$ is surjective for some/all $s \in \mathbb{C}^+$, see Lemma 40 in the appendix.

$b) \Rightarrow c)$: This holds trivially, since when $sI - \mathscr{A}_{red}$ is boundedly invertible, its range equals $\mathbb{X}_1$ and the operator is closed.

$c) \Rightarrow a)$: We begin by showing that $s\mathscr{E}_I - \mathscr{A}$ is injective. Let $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in D(\mathscr{A})$ be such that $(s\mathscr{E}_I - \mathscr{A}) \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = 0$. So

$$0 = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, (s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = \bar{s}\|x_1\|^2 - \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathscr{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle$$

where the last term has nonnegative real part, since $\mathscr{A}$ is dissipative. If $x_1 \neq 0$, the first part would have strictly positive real part, which contradicts the equality and thus $x_1 = 0$. This gives that

$$0 = (s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = -\mathscr{A} \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$

and in particular $\mathscr{A}_2 \left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right] = 0$, which by assumption gives $x_2 = 0$. Thus we have shown that $s\mathscr{E}_I - \mathscr{A}$ is injective.

Next we prove the surjectivity. Let $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in \mathbb{X}$ be given. By the surjectivity of $\mathscr{A}_2$ there exists an $\left[\begin{smallmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{smallmatrix}\right] \in D(\mathscr{A})$ such that $\mathscr{A}_2 \left[\begin{smallmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{smallmatrix}\right] = -y_2$. Defining

$$\tilde{y}_1 = s\tilde{x}_1 - \mathscr{A}_1 \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix},$$

we obtain

$$\begin{bmatrix} \tilde{y}_1 \\ y_2 \end{bmatrix} = (s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}. \tag{45}$$

Since $(sI - \mathscr{A}_{red})$ is surjective, there exists x_1 such that $y_1 - \tilde{y}_1 = (sI - \mathscr{A}_{red})x_1$. By the definition of $\mathscr{A}_{red}$ this means that there exists x_2 such

$$\begin{bmatrix} y_1 - \tilde{y}_1 \\ 0 \end{bmatrix} = (s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{46}$$

Adding (45) and (46) gives that $\left[\begin{smallmatrix} x_1 + \tilde{x}_1 \\ x_2 + \tilde{x}_2 \end{smallmatrix}\right]$ is mapped to $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ by $s\mathscr{E}_I - \mathscr{A}$, and we conclude that $s\mathscr{E}_I - \mathscr{A}$ is surjective.

Since $\mathscr{A} - s\mathscr{E}_I$ is injective and surjective and since $\mathscr{A}$ is closed, so is $\mathscr{A} - s\mathscr{E}_I$ and thus injectivity and surjectivity implies bounded invertibility, see e.g. [Corollary A.3.50][9]. □

In the part of the proof of Lemma 26 that c) implies d) we show that we only needed the closedness of $\mathscr{A}$ to conclude from the injectivity plus surjectivity the bounded invertibility of $s\mathscr{E}_I - \mathscr{A}_{red}$. Since $\mathscr{A} - s\mathscr{E}_I$ is dissipative, it is closable, see Section 7. The closure is obviously still surjective, and thus it remains to show that it is injective. The $s\mathscr{E}_I$ term gives that any element in the kernel should have $x_1 = 0$. If the following implication holds: $\mathscr{A}_2 \left[\begin{smallmatrix} 0 \\ x_{2,n} \end{smallmatrix}\right] \to 0$, $x_{2,n} \to x_2 \Rightarrow x_2 = 0$, then the closure is injective.

Theorem 25 implies that for all $x_{1,0} \in \mathbb{X}_1$ there exists a unique (weak or mild) solution of

$$\dot{x}_1(t) = \mathscr{A}_{red} x_1(t), \quad x_1(0) = x_{1,0}. \tag{47}$$

However, this is only a part of the solution of the adHDAE (1). For a classical solution, we have that $x_1(t) \in D(\mathscr{A}_{red})$, and so since a given x_1 yields a unique x_2, we also find a (unique) $x_2(t)$ such that the bottom equation of (1) is satisfied. In general the equation for $x_2(t)$ will not exist for all mild solutions, as is shown on basis of Example 28, see the text following that example.

4 Applications

In this section we study several well-known classes of systems, and show that they can be seen as examples of Theorem 25. We start with the class of abstract port-Hamiltonian systems.

4.1 Abstract port-Hamiltonian systems on a 1D spatial domain.

In this section we discuss port-Hamiltonian systems and we begin with a very general setup. Let L^2, H^1, H^2 denote the usual Hilbert spaces of square integrable functions, and associated Sobolev spaces. On $L^2((0,1);\mathbb{R}^n)$ we consider the operator

$$\mathscr{A}x = P_1 \frac{d}{d\zeta} x + G_0(\zeta) x \tag{48}$$

with domain

$$D(\mathscr{A}) = \{x \in H^1((0,1);\mathbb{R}^n) \mid W_B \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = 0\}. \tag{49}$$

Here P_1 is a real constant, symmetric, invertible matrix, and $G_0 : [0,1] \mapsto \mathbb{C}^{n\times n}$ is Lipschitz continuous satisfying $G_0(\zeta)+G_0(\zeta)^* \leq 0$ for all $\zeta \in [0,1]$. Furthermore, W_B is a (constant) $n \times 2n$ matrix of full rank. From [23, 27] or [22] it is known that $\mathscr{A}$ is maximally dissipative if and only if

$$v^T P_1 v - w^T P_1 w \leq 0 \quad \text{for all } v,w \in \mathbb{R}^n \text{ satisfying } W_B \begin{bmatrix} v \\ w \end{bmatrix} = 0. \tag{50}$$

For this class of systems we show that if the conditions of Lemma 26 hold, then the associated operator pair is regular.

Theorem 27 *Consider the adHDAE system (1), where the operator $\mathscr{A}$ has its domain defined in equations (48) and (49). Furthermore, assume that (50) holds. Let $n_1+n_2 = n$ and write $\mathbb{X} = L^2((0,1);\mathbb{R}^n) = L^2((0,1);\mathbb{R}^{n_1}) \oplus L^2((0,1);\mathbb{R}^{n_2}) =: \mathbb{X}_1 \oplus \mathbb{X}_2$.*

If the subset $\mathbb{V}_0 := \{x_2 \in \mathbb{X}_2 \mid \left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right] \in D(\mathscr{A}) \text{ and } \mathscr{A}_2 \left[\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right] = 0\}$ contains only the zero element, then $(\mathscr{E}_I, \mathscr{A})$ is regular, i.e. Assumption 9 is satisfied for this class of systems.

If the $n_2 \times n_2$ right lower block of P_1 is zero and the corresponding block of $G_0(\zeta)$ is invertible for almost all $\zeta \in [0,1]$, then $\mathbb{V}_0 = \{0\}$.

Proof. We have to show that $s\mathscr{E}_I - \mathscr{A}$ is boundedly invertible. To do so we introduce some notation. We split the matrices according to the dimensions n_1 and n_2, i.e.

$$P_1 = \begin{bmatrix} P_{1,11} & P_{1,12} \\ P_{1,21} & P_{1,22} \end{bmatrix} \quad G_0 = \begin{bmatrix} G_{0,11} & G_{0,12} \\ G_{0,21} & G_{0,22} \end{bmatrix}. \tag{51}$$

The equation $y = (\mathscr{E}_I - \mathscr{A})x$ can then equivalently be written as

$$P_1 \frac{dx}{d\zeta}(\zeta) = \begin{bmatrix} sI - G_{0,11}(\zeta) & -G_{0,12}(\zeta) \\ -G_{0,21}(\zeta) & -G_{0,22}(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta) \\ x_2(\zeta) \end{bmatrix} - \begin{bmatrix} y_1(\zeta) \\ y_2(\zeta) \end{bmatrix} =: -G_s(\zeta)x(\zeta) - y(\zeta).$$

Since P_1 is invertible, this is an implicit linear ordinary differential equation in ζ with variable coefficients. Since G_s is Lipschitz continuous, for every initial condition $x(0)$ this equation has a unique solution, which we write as

$$x(\zeta) = \begin{bmatrix} x_1(\zeta) \\ x_2(\zeta) \end{bmatrix} = \Psi(\zeta,0) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^\zeta \Psi(\zeta,\tau) \begin{bmatrix} -y_1(\tau) \\ -y_2(\tau) \end{bmatrix} d\tau,$$

where Ψ is the fundamental solution matrix of the homogeneous system. If we would have that this solution is in the domain of $\mathscr{A}$, then this part of the proof is complete. For this we need that

$$W_B \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = 0,$$

or equivalently

$$(W_{B,1}\Psi(1,0) + W_{B,2})x(0) = W_{B,1} \int_0^1 \Psi(1,\tau) \begin{bmatrix} y_1(\tau) \\ y_2(\tau) \end{bmatrix} d\tau.$$

If the (constant) matrix in front of $x(0)$ is invertible, then we can find a unique $x(0)$, and so the solution $x(\cdot)$ is uniquely determined. If this matrix is not invertible, then choose $0 \neq x(0)$ in its kernel, i.e.,

$$(W_{B,1}\Psi(1,0) + W_{B,2})x(0) = 0.$$

This implies that

$$x(\zeta) := \Psi(\zeta,0)x(0)$$

is a solution of $P_1 \frac{dx}{d\zeta}(\zeta) = -G_s(\zeta)x(\zeta)$ which satisfies the boundary condition, i.e. $W_B \left[\begin{smallmatrix} x(1) \\ x(0) \end{smallmatrix}\right] = 0$. By the definition of G_s, this means that x satisfies $(s\mathscr{E}_I - \mathscr{A})x = 0$, implying that $s\mathscr{E}_I - \mathscr{A}$ is not injective.

By Lemma 18 this means that the first component of x is zero, and the second component satisfies $\mathscr{A}\left[\begin{smallmatrix}0\\x_2\end{smallmatrix}\right]=0$. By our assumption this gives that $x_2=0$. Concluding, we see that $(W_{B,1}\Psi(1,0)+W_{B,2})$ must be injective and thus surjective, implying that the pair $(\mathscr{E}_I,\mathscr{A})$ is regular.

To prove the last statement, considering (51), we get

$$\mathscr{A}_2\begin{bmatrix}0\\x_2\end{bmatrix}=\begin{bmatrix}P_{1,12}\frac{dx_2}{d\zeta}+G_{0,12}x_2\\G_{0,22}x_2\end{bmatrix}, \tag{52}$$

where we have used the condition on P_1. Using the invertibility of $G_{0,22}$ this can only be zero, when $x_2=0$. □

We see from (52) that even when $G_{0,22}$ is singular, this equation could have only the zero function as its solution. Since P_1 is invertible and $P_{1,22}=0$, $P_{1,12}$ is of full rank and so $\mathscr{A}\left[\begin{smallmatrix}0\\x_2\end{smallmatrix}\right]=0$ is (partly) a differential equation. This means that it will also depend on the boundary conditions, imposed by W_B, whether $x_2=0$ is its only solution.

There are several applications of Theorem 27.

Example 28 *Choose*

$$P_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}\text{ and }G_0=\begin{bmatrix}-g_0&0\\0&-r\end{bmatrix},$$

where g_0 is a bounded function and r is a bounded and invertible function, and moreover both satisfy that their real part is non-negative. Furthermore, choose

$$\mathscr{E}=\begin{bmatrix}e_1&0\\0&0\end{bmatrix}\text{ and }\mathscr{Q}=\begin{bmatrix}1&0\\0&q_2\end{bmatrix},$$

where e_1,q_2 are positive, bounded, and invertible functions. We take a full rank W_B such that (50) holds, and thus $\mathscr{A}$ is dissipative. For $n_1=n_2=1$, it is not hard to see that the assumptions of Theorem 27 are satisfied. Hence $(\mathscr{E}_I,\mathscr{A})$ is regular, and so is $(\mathscr{E},\mathscr{A}\mathscr{Q})$, see Corollary 15.

Applying Theorem 25, by (43) we find that

$$\mathscr{A}_{red}x_1=\frac{1}{e_1}\left[\frac{d(q_2x_2)}{d\zeta}-g_0x_1\right]\text{ with }\frac{dx_1}{d\zeta}-rq_2x_2=0$$

or equivalently

$$(\mathscr{A}_{red}x_1)(\zeta)=\frac{1}{e_1(\zeta)}\left[\frac{d}{d\zeta}\left(\frac{1}{r(\zeta)}\frac{dx_1}{d\zeta}(\zeta)\right)-g_0(\zeta)x_1(\zeta)\right] \tag{53}$$

with domain

$$D(\mathscr{A}_{red})=\{x_1\in H^1(0,1)\mid \frac{1}{r}\frac{dx_1}{d\zeta}\in H^1(0,1)\text{ and }W_B\begin{bmatrix}x_1(1)\\\frac{1}{r(1)}\frac{dx_1}{d\zeta}(1)\\x_1(0)\\\frac{1}{r(0)}\frac{dx_1}{d\zeta}(0)\end{bmatrix}=0\}.$$

If we assume that r is real-valued, then this operator is (minus) a Sturm-Liouville operator, see [9, p. 82], with the exception of the sign condition on the last term. This sign condition is a consequence of the fact that we want dissipative operators, whereas that is not imposed in general for Sturm-Liouville operators.

Sturm-Liouville operators always come with a specific set of boundary conditions. We can obtain these boundary conditions by choosing the right W_B, e.g. with the real matrix

$$W_B=\begin{bmatrix}\alpha_1&\beta_1&0&0\\0&0&\alpha_2&\beta_2\end{bmatrix},$$

with $\alpha_1^2+\beta_1^2>0$ and $\alpha_2^2+\beta_2^2>0$. This matrix satisfies (50) if and only if $\alpha_1\beta_1\geq 0$ and $\alpha_2\beta_2\geq 0$. Again these conditions are a consequence of the fact that we want dissipative operators, whereas that is not imposed for Sturm-Liouville operators.

The diffusion/heat equation is the most well-known Sturm-Liouville operator. If we choose $r=e_1=1$, $\alpha_1=\alpha_2=1$ and $\beta_1=\beta_2=0$, then the PDE $\dot{x}(t)=\mathscr{A}x(t)$ corresponds to an undamped vibrating string which is fixed at the boundary, whereas the PDE $\dot{x}_1(t)=\mathscr{A}_{red}x_1(t)$ corresponds to the diffusion/heat equation with temperature zero at the boundary.

So we have constructed the heat equation out of the wave equation. If we choose $r=-i$, then the PDE $\dot{x}_1(t)=\mathscr{A}_{red}x_1(t)$ corresponds to the 1-D Schrödinger equation.

Now we return to the comments made below equation (47). We once more look at the differential equation we found for x_1 in Example 28. For simplicity, we assume that $e_1=1$, $r=-i$, and so x_1 satisfies the standard Schrödinger equation. It is well-known that for an arbitrary initial condition in $L^2(0,1)$ this will have a unique weak/mild solution. However, for an initial condition in $L^2(0,1)$ the solution will not get smoother, and so $x_2(t)=i\frac{\partial}{\partial\zeta}x_1(t)$ will in general not lie in the state space.

Next we apply Theorem 27 to show that the equations for an Euler-Bernoulli beam can be constructed out of two wave equations.

Example 29 *Consider $\mathscr{A}$ of equation (48) with*

$$P_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\quad G_0=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$$

and assume that W_B is a full rank 4×8 matrix satisfying (50). We take $n_1=n_2=2$, $\mathscr{E}=\operatorname{diag}(\mathscr{E}_1,0)$, $\mathscr{Q}=\operatorname{diag}(\mathscr{Q}_1,\mathscr{Q}_2)$, with $\mathscr{E}_1,\mathscr{Q}_1,\mathscr{Q}_2$ strictly positive (2×2)-matrix valued bounded functions. It is easy to see that the conditions of Theorem 27 are satisfied, and so are those of Assumption 9.

The operator $\mathscr{A}_{red}$ from Theorem 25 then becomes

$$\mathscr{A}_{red}x_1=\mathscr{E}_1^{-1}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{d}{d\zeta}\left(\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{d(\mathscr{Q}_1x_1)}{d\zeta}\right)=\mathscr{E}_1^{-1}\begin{bmatrix}0&-1\\1&0\end{bmatrix}\frac{d^2(\mathscr{Q}_1x_1)}{d\zeta^2}. \tag{54}$$

For $\mathscr{E}_1=\left[\begin{smallmatrix}\rho&0\\0&1\end{smallmatrix}\right]$, $\mathscr{Q}_1=\left[\begin{smallmatrix}q_1&0\\0&q_2\end{smallmatrix}\right]$, and $x_1:=\left[\begin{smallmatrix}x_{1,1}\\x_{1,2}\end{smallmatrix}\right]$ the associated PDE $\dot{x}_1(t)=\mathscr{A}_{red}x_1(t)$ takes the form

$$\frac{\partial x_{1,1}}{\partial t}=-\frac{1}{\rho}\frac{\partial^2(q_2x_{1,2})}{\partial\zeta^2}\quad\text{and}\quad\frac{\partial x_{1,2}}{\partial t}=\frac{\partial^2(q_1x_{1,1})}{\partial\zeta^2},$$

or in the variable $x_{1,1}$

$$\rho(\zeta)\frac{\partial^2x_{1,1}}{\partial t^2}(\zeta,t)=-\frac{\partial^2}{\partial\zeta^2}\left[q_2(\zeta)\frac{\partial^2(q_1(\zeta)x_{1,1})}{\partial\zeta^2}(\zeta,t)\right].$$

For ρ the mass density, $q_1=1$, and $q_2=EK$, with E the elastic modulus and K the second moment of area of the beam's cross section, this is the well-known Euler-Bernoulli beam model.

We note that P_1 can be seen to correspond to two wave equations, namely one in the variables $x_{1,1}$ and $x_{1,4}$ and the other in the variables $x_{1,2}$ and $x_{1,3}$. So we can construct the beam equation out of two wave equations.

In Example 29 we have discussed the construction of a second order port-Hamiltonian system from a first order one, see also [27]. In the following lemma we will do this generally, and also pay attention to the boundary conditions.

Lemma 30 *Consider an operator $\mathscr{A}_{red}$ on $L^2((0,1);\mathbb{R}^{n_1})$ of the form*

$$\mathscr{A}_{red}x_1 = P_2\frac{d^2x_1}{d\zeta^2} + P_{1,1}\frac{dx_1}{d\zeta} + P_0x_1$$

with domain

$$D(\mathscr{A}_{red}) = \{x_1 \in H^2((0,1);\mathbb{R}^{n_1}) \mid \tilde{W}_B \begin{bmatrix} x_1(1) \\ \frac{dx_1}{d\zeta}(1) \\ x_1(0) \\ \frac{dx_1}{d\zeta}(0) \end{bmatrix} = 0\},$$

where we assume that, for the $n_1 \times n_1$ coefficient matrices, we have that P_2, P_0 are skew-symmetric, $P_{1,1}$ is symmetric and P_2 is invertible. Furthermore, $\tilde{W}_B$ is a full rank $2n_1 \times 4n_1$-matrix.

If $\mathscr{A}_{red}$ is a generator of a contraction semigroup on $L^2((0,1);\mathbb{R}^{n_1})$, then it can be constructed via Theorem 25 from an $\mathscr{A}$ as in (48) and (49).

Proof. We recall from [27] that under the conditions on the coefficient matrices, $\mathscr{A}_{red}$ generates a contraction semigroup if and only if

$$\begin{bmatrix} v_{1,1}^T & v_{1,2}^T \end{bmatrix} \begin{bmatrix} P_{1,1} & P_2 \\ -P_2 & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} - \begin{bmatrix} w_{1,1}^T & w_{1,2}^T \end{bmatrix} \begin{bmatrix} P_{1,1} & P_2 \\ -P_2 & 0 \end{bmatrix} \begin{bmatrix} w_{1,1} \\ w_{1,2} \end{bmatrix} \leq 0 \tag{55}$$

for $v_{1,1}, v_{1,2}, w_{1,1}, w_{1,2} \in \mathbb{R}^{n_1}$ such that $\tilde{W}_B \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ w_{1,1} \\ w_{1,2} \end{bmatrix} = 0$.

With the matrices $P_2, P_{1,1}$, and P_0 we choose the following $n \times n = 2n_1 \times 2n_1$-matrices in (48)

$$P_1 = \begin{bmatrix} P_{1,1} & I \\ I & 0 \end{bmatrix} \text{ and } G_0 = \begin{bmatrix} P_0 & 0 \\ 0 & -P_2^{-1} \end{bmatrix}. \tag{56}$$

From our assumption and choices we see that $P_1^T = P_1$ and $G_0^T = -G_0$. Furthermore, P_1 is invertible.

Next we choose the matrix W_B in (49) as

$$W_B = \tilde{W}_B \cdot \operatorname{diag}(I, P_2^{-1}, I, P_2^{-1}). \tag{57}$$

It is clear that this has full rank $n = 2n_1$. It remains to check that for these choices the operator $\mathscr{A}$ with domain $D(\mathscr{A})$, defined via (48) and (49), satisfy the condition (50).

First we note that $\begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} \in \ker W_B$ if and only if $\begin{bmatrix} v_1 \\ P_2^{-1}v_2 \\ w_1 \\ P_2^{-1}w_2 \end{bmatrix} \in \ker \tilde{W}_B$. Secondly, the following equality holds

$$\begin{aligned} \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} P_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} \begin{bmatrix} P_{1,1} & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} \begin{bmatrix} P_{1,1} & P_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ P_2^{-1}v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1^T & (P_2^{-1}v_2)^T \end{bmatrix} \begin{bmatrix} P_{1,1} & P_2 \\ -P_2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ P_2^{-1}v_2 \end{bmatrix}. \end{aligned} \tag{58}$$

Combining these two facts with (55) we have that

$$\begin{bmatrix} v_1^T & v_2^T \end{bmatrix} P_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1^T & w_2^T \end{bmatrix} P_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \leq 0 \text{ for all } \begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} \in \ker W_B.$$

So we have that the operator $\mathscr{A}$ defined in (48) and 49), with P_1 and G_0 given in (56), is maximally dissipative. Choosing $n_2 = n_1$, we see that all the conditions needed in Theorem 27 are satisfied.

Choosing $\mathscr{E} = \mathscr{E}_I$ and $\mathscr{Q} = I$, the operator from (43) is given as

$$\begin{bmatrix} P_{1,1} & I \end{bmatrix} \frac{d}{d\zeta} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + P_0 x_1 \text{ with } \frac{dx_1}{d\zeta} - P_2^{-1} x_2 = 0, \tag{59}$$

with domain

$$\{x_1, x_2 \in H^1((0,1);\mathbb{R}^{n_1}) \mid W_B \begin{bmatrix} x_1(1) \\ x_2(1) \\ x_1(0) \\ x_2(0) \end{bmatrix} = 0\}. \tag{60}$$

From (59) we obtain $x_2 = P_2 \frac{dx_1}{d\zeta}$. Substituting this in the first equality of (59) and in (60) gives the operator $\mathscr{A}_{red}$ and its domain as asserted. □

We have shown how different models can be constructed out of the wave equation model by imposing a closure relation. This is the opposite construction as is usually done in Stokes or Oseen equations where the heat equation is obtain by a restriction, see [12, 43].

In the following example we show that we can as well obtain coupled PDEs which act on different physical domains.

Example 31 *Consider the operator $\mathscr{A}$ of equation (48) with*

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \end{bmatrix}$$

with r is a bounded and invertible function satisfying $Re(r(\zeta)) \geq 0$ for all $\zeta \in [0,1]$. We choose its domain to be given by (49) with

$$W_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is clear that this is of full rank, and it is not hard to see that (50) holds. We take $n_1 = 3$, $n_2 = 1$. With these choices it is straightforward to see that the conditions of Theorem 27 hold, and so $(\mathscr{E}_I, \mathscr{A})$ is regular. Using Corollary 15, we can build the operator $\mathscr{A}_{red}$ from Theorem 25. For this we choose $\mathscr{E} = \mathscr{E}_I$, and $\mathscr{Q} = \operatorname{diag}(\rho^{-1}, T, 1, 0)$ with ρ, T (strictly) positive functions. This operator then satisfies

$$\mathscr{A}_{red} x_1 = \begin{bmatrix} \frac{d(Tx_{1,2})}{d\zeta} \\ \frac{d(\rho^{-1}x_{1,1})}{d\zeta} \\ \frac{dx_2}{d\zeta} \end{bmatrix} \text{ with } \frac{dx_{1,3}}{d\zeta} = rx_2.$$

The corresponding PDE splits into the two PDEs

$$\frac{\partial}{\partial t} \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \rho^{-1} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} \right) \text{ and } \frac{\partial x_{1,3}}{\partial t} = \frac{\partial}{\partial \zeta} \left[r^{-1} \frac{\partial x_{1,3}}{\partial \zeta} \right].$$

In the first PDE we recognise the wave equation, whereas the second is a heat/diffusion equation. They seem to be uncoupled, but we have not looked at the boundary conditions of $\mathscr{A}$. Using the closure relation $\frac{dx_{1,3}}{d\zeta} = rx_2$, we see that the boundary conditions become

$$\rho^{-1} x_{1,1}(1) = x_{1,3}(0), \; Tx_{1,2}(1) = r^{-1} \frac{\partial x_{1,3}}{\partial \zeta}(0), \; x_{1,1}(0) = 0 \text{ and } \frac{\partial x_{1,3}}{\partial \zeta}(1) = 0.$$

So in this way the heat equation is coupled at the boundary to the wave equation, but certainly other couplings are possible as well.

The proof that $\mathscr{A}$ with $D(\mathscr{A})$ given in (48) and (49) is maximally dissipative if and only if (50) holds was given in [27] by using boundary triplets, which will be the topic of the next subsection.

4.2 Boundary triplets

In this section we illustrate that our approach can also be used in the context of boundary triplets to derive results that have also been obtained via different approaches. We begin by recalling the concept of a *boundary triplet*. Let $\mathscr{A}_m$ be a densely defined operator on a Hilbert space $\mathscr{H}$ with dual $\mathscr{A}_m^*$, and let Γ_1, Γ_2 be two linear mappings from $D(\mathscr{A}_m^*)$ to another Hilbert space $\mathbb{U}$. The triplet $(\mathbb{U}, \Gamma_1, \Gamma_2)$ is a *boundary triplet* if the following conditions are satisfied, see [17, section 3.1]:

1. For all $f, g \in D(\mathscr{A}_m^*)$ it holds that
$$\langle \mathscr{A}_m^* f, g\rangle_{\mathscr{H}} - \langle f, \mathscr{A}_m^* g\rangle_{\mathscr{H}} = \langle \Gamma_1 f, \Gamma_2 g\rangle_{\mathbb{U}} - \langle \Gamma_2 f, \Gamma_1 g\rangle_{\mathbb{U}}. \tag{61}$$
2. For all $u_1, u_2 \in \mathbb{U}$ there exists $f \in D(\mathscr{A}_m^*)$ such that $\Gamma_1 f = u_1$ and $\Gamma_2 f = u_2$.

By choosing f in the kernel of the boundary operators Γ_1 and Γ_2, we see that the corresponding restriction of $\mathscr{A}_m^*$ is symmetric, and not skew-symmetric, as is normally the case for generators of contraction semigroups. Therefore we will work with $i\mathscr{A}_m^*$ and $i\Gamma_1, \Gamma_2$.

For these operators, (61) becomes (equivalently)

$$\langle i\mathscr{A}_m^* f, g\rangle_{\mathscr{H}} + \langle f, i\mathscr{A}_m^* g\rangle_{\mathscr{H}} = \langle i\Gamma_1 f, \Gamma_2 g\rangle_{\mathbb{U}} + \langle \Gamma_2 f, i\Gamma_1 g\rangle_{\mathbb{U}}. \tag{62}$$

In Theorem 3.1.6 of [17] it is shown that $i\mathscr{A}_m^*$ restricted to the domain

$$\{x_0 \in D(\mathscr{A}_m^*) \mid (\mathscr{K} - I)\Gamma_1 x_0 + i(\mathscr{K} + I)\Gamma_2 x_0 = 0\} \tag{63}$$

with $\mathscr{K}$ satisfying $\|\mathscr{K}\| \leq 1$, are all maximally dissipative restrictions of $i\mathscr{A}_m^*$. We will show that this result can be obtained alternatively via Theorem 25.

To show this, for a given boundary triplet, we define $\mathbb{X}_1 = \mathscr{H}$, $\mathbb{X}_2 = \mathbb{U}$, and

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \end{bmatrix} = \begin{bmatrix} \mathscr{A}_1 & \\ \frac{1}{2}L(-i\Gamma_1 + \Gamma_2) & -\frac{1}{2}I \end{bmatrix} \tag{64}$$

with

$$\mathscr{A}_1 \begin{bmatrix} x_1 \\ u \end{bmatrix} = i\mathscr{A}_m^* x_1,$$
$$D(\mathscr{A}) = \left\{ \begin{bmatrix} x_1 \\ u \end{bmatrix} \mid x_1 \in D(\mathscr{A}_m^*) \text{ with } (i\Gamma_1 + \Gamma_2)x_1 = u \right\}, \tag{65}$$

and $L \in \mathscr{L}(\mathbb{U})$.

Next we study the dissipativity of $\mathscr{A}$. By the definition of $\mathscr{A}$ and relation (62) we find

$$\begin{aligned}
\langle \mathscr{A} \begin{bmatrix} x_1 \\ u \end{bmatrix}, \begin{bmatrix} x_1 \\ u \end{bmatrix}\rangle + \langle \begin{bmatrix} x_1 \\ u \end{bmatrix}, \mathscr{A} \begin{bmatrix} x_1 \\ u \end{bmatrix}\rangle& \\
&= \langle i\mathscr{A}_m^* x_1, x_1\rangle + \langle x_1, i\mathscr{A}_m^* x_1\rangle + \\
&\quad \langle \frac{1}{2}L(-i\Gamma_1 + \Gamma_2)x_1, u\rangle_{\mathbb{U}} + \langle u, \frac{1}{2}L(-i\Gamma_1 + \Gamma_2)x_1\rangle_{\mathbb{U}} \\
&\quad - \langle \frac{1}{2}u, u\rangle_{\mathbb{U}} - \langle u, \frac{1}{2}u\rangle_{\mathbb{U}} \\
&= \langle i\Gamma_1 x_1, \Gamma_2 x_1\rangle_{\mathbb{U}} + \langle \Gamma_2 x_1, i\Gamma_1 x_1\rangle_{\mathbb{U}} + \\
&\quad \langle \frac{1}{2}L(-i\Gamma_1 + \Gamma_2)x_1, u\rangle_{\mathbb{U}} + \langle u, \frac{1}{2}L(-i\Gamma_1 + \Gamma_2)x_1\rangle_{\mathbb{U}} - \langle u, u\rangle_{\mathbb{U}}.
\end{aligned}$$

Now we define $y = (-i\Gamma_1 + \Gamma_2)x_1$, and using (65) it is easy to see that

$$\langle i\Gamma_1 x_1, \Gamma_2 x_1\rangle_{\mathbb{U}} + \langle \Gamma_2 x_1, i\Gamma_1 x_1\rangle_{\mathbb{U}} = \frac{1}{2}\langle u,u\rangle_{\mathbb{U}} - \frac{1}{2}\langle y,y\rangle_{\mathbb{U}}.$$

Hence

$$\begin{aligned}\langle \mathscr{A}\begin{bmatrix} x_1 \\ u \end{bmatrix}, \begin{bmatrix} x_1 \\ u \end{bmatrix}\rangle + \langle \begin{bmatrix} x_1 \\ u \end{bmatrix}, \mathscr{A}\begin{bmatrix} x_1 \\ u \end{bmatrix}\rangle & \\ &= \frac{1}{2}\langle u,u\rangle_{\mathbb{U}} - \frac{1}{2}\langle y,y\rangle_{\mathbb{U}} + \\ &\langle \frac{1}{2}Ly, u\rangle_{\mathbb{U}} + \langle u, \frac{1}{2}Ly\rangle_{\mathbb{U}} - \langle u,u\rangle_{\mathbb{U}} \\ &= \langle \begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}L \\ \frac{1}{2}L^* & -\frac{1}{2}I \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} u \\ y \end{bmatrix}\rangle_{\mathbb{U}\oplus\mathbb{U}}. \end{aligned} \tag{66}$$

Using the equality

$$\begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}L \\ \frac{1}{2}L^* & -\frac{1}{2}I \end{bmatrix} = \begin{bmatrix} I & -L \\ 0 & I \end{bmatrix}\begin{bmatrix} -\frac{1}{2}I + \frac{1}{2}LL^* & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix}\begin{bmatrix} I & 0 \\ -L^* & I \end{bmatrix}$$

and (66) we see that the operator $\mathscr{A}$ is dissipative if and only if $LL^* \leq I$, or equivalently if $\|L\| \leq 1$. Next we choose $\mathscr{E} = \mathscr{E}_I$ and $\mathscr{Q} = I$, and so for $\|L\| \leq 1$ all conditions in Assumption 9 are satisfied except possibly the regularity. By Lemma 26, the regularity can be checked by the maximally dissipativity of $\mathscr{A}_{red}$, the closedness of $\mathscr{A}$, and $\mathscr{A}_2$ being surjective. Since the pair (Γ_1, Γ_2) is surjective, it follows that for every $u \in \mathbb{U}$ there exists an $x_1 \in D(\mathscr{A}_m^*)$ such that $(-i\Gamma_1 + \Gamma_2)x_1 = 0$ and $(i\Gamma_1 + \Gamma_2)x_1 = -2u$. Hence $\left[\begin{smallmatrix} x_1 \\ -2u \end{smallmatrix}\right] \in D(\mathscr{A})$, and $\mathscr{A}_2\left[\begin{smallmatrix} x_1 \\ -2u \end{smallmatrix}\right] = u$, and thus $\mathscr{A}_2$ is surjective. That $\mathscr{A}$ is closed follows from the fact that $\mathscr{A}_m^*$ is closed.

So to obtain the regularity, we have to study the operator $\mathscr{A}_{red}$. Note that the definition of the domain of $\mathscr{A}_{red}$ already gives that the condition $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right] \in D(\mathscr{A}_{red})$ implies that $u = 0$. So all conditions of Theorem 25 are satisfied.

We find that $\mathscr{A}_{red}$ is given via

$$\mathscr{A}_{red}x_1 = i\mathscr{A}_m^* x_1$$

with domain

$$\begin{aligned} D(\mathscr{A}_{red}) &= \{x_1 \in D(\mathscr{A}_m^*) \mid (i\Gamma_1 + \Gamma_2)x_1 = u = L(-i\Gamma_1 + \Gamma_2)x_1\} \\ &= \{x_1 \in D(\mathscr{A}_m^*) \mid (L+I)i\Gamma_1 x_1 + (-L+I)\Gamma_2 x_1 = 0\}. \end{aligned}$$

Multiplying this expression with i and taking $\mathscr{K} = -L$ we obtain (63), i.e., the condition of [17].

To complete the regularity proof it remains to show that $\mathscr{A}_{red}$ is maximally dissipative which is shown in [17].

In this subsection we have seen that boundary triplets fit into the framework of adHDAEs and in the next subsection we show this for impedance passive systems.

4.3 Impedance passive systems

Let $\mathscr{H}$, $\mathbb{V}$, and $\mathbb{U}$ be Hilbert spaces and let $\left[\begin{smallmatrix} L \\ K_0 \end{smallmatrix}\right]$ be a closed operator from $\mathbb{V}$ to $\mathscr{H} \oplus \mathbb{U}$. We define $\mathbb{V}_0 := D\left(\left[\begin{smallmatrix} L \\ K_0 \end{smallmatrix}\right]\right) \subset \mathbb{V}$. Since $\left[\begin{smallmatrix} L \\ K_0 \end{smallmatrix}\right]$ is closed, $\mathbb{V}_0$ with its graph norm is a Hilbert space and $\left[\begin{smallmatrix} L \\ K_0 \end{smallmatrix}\right]$ is a bounded operator from $\mathbb{V}_0$ to $\mathscr{H} \oplus \mathbb{U}$. Therefore L^* and K_0^* are in $\mathscr{L}(\mathscr{H}, \mathbb{V}_0^*)$ and $\mathscr{L}(\mathbb{U}, \mathbb{V}_0^*)$, respectively. We view $\mathbb{V}$ as the pivot space, i.e., $\mathbb{V}_0 \hookrightarrow \mathbb{V} = \mathbb{V}^* \hookrightarrow \mathbb{V}_0^*$ are subsets with dense continuous injections.

Motivated by the Maxwell equation as well as the (damped) beam equation, the following system was introduced in [42].

$$\dot{x}(t) = \begin{bmatrix} 0 & -L \\ L^* & G - K_0^* K_0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \sqrt{2} K_0^* \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0 & -\sqrt{2} K_0 \end{bmatrix} x(t), \tag{67}$$

on the state space $\mathbb{X}_1 = \mathscr{H} \oplus \mathbb{V}$. Here L, K_0 satisfy the properties stated above, and $G \in \mathscr{L}(\mathbb{V}_0, \mathbb{V}_0^*)$. With our notation, we see how we have to interpret (67). Namely, the system operator has the following domain

$$D\left(\left[\begin{smallmatrix} 0 & -L \\ L^* & G - K_0^* K_0 \end{smallmatrix}\right]\right) = \left\{ \begin{bmatrix} h \\ v \end{bmatrix} \in \mathscr{H} \oplus \mathbb{V}_0 \mid L^* h + (G - K_0^* K_0) v \in \mathbb{V} \right\}, \tag{68}$$

where the addition is done in $\mathbb{V}_0^*$. For the rest of this subsection we concentrate on this system operator.

For the study of the system in [42] the following operator is introduced

$$\mathscr{T} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & 0 \end{bmatrix} \tag{69}$$

with domain

$$D(\mathscr{T}) = \left\{ \begin{bmatrix} h \\ e \\ u \end{bmatrix} \in \mathscr{H} \oplus \mathbb{V}_0 \oplus \mathbb{U} \mid L^* h + Ge + K_0^* u \in \mathbb{V} \right\}. \tag{70}$$

In [42] it is shown that $\mathscr{T}$ is maximally dissipative[1] if

$$\operatorname{Re}\langle Ge, e \rangle_{\mathbb{V}_0^*, \mathbb{V}_0} \leq 0. \tag{71}$$

Under this condition, they apply an "external Cayley transform" to show that the system (67) is well-defined. This gives that the system operator generates a contraction semigroup, and thus is maximally dissipative. We will show that this result can also be obtained via our techniques. For this we define $\mathbb{X}_2 = \mathbb{U}$, and

$$\mathscr{A} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & -I \end{bmatrix}$$

with the domain given by that of $\mathscr{T}$, see (70). Since $\mathscr{A}$ differs from $\mathscr{T}$ by just the $-I$ is the lower right corner it is also maximally dissipative when (71) holds. We choose $\mathscr{E} = \mathscr{E}_I$ and $\mathscr{Q} = I$. Since $\mathscr{T}$ is maximally dissipative, we have that $\mathscr{A} + \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ is maximally dissipative, and so by Lemma 16 $(\mathscr{E}, \mathscr{A}\mathscr{Q})$ is regular. Again by the $-I$ is the lower right corner of $\mathscr{A}$, we see that the condition of Theorem 25 is satisfied, and thus the operator $\mathscr{A}_{red}$ defined by

$$\mathscr{A}_{red} x_1 = \mathscr{A}_{red} \begin{bmatrix} h \\ e \end{bmatrix} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \end{bmatrix} \begin{bmatrix} h \\ e \\ u \end{bmatrix} \text{ with } -K_0 e - u = 0$$

generates a contraction semigroup. It is now straightforward to see that this is the system operator from (67) and (68). So applying Theorem 25 we obtain the result of [42].

In [42] a similar result is also obtained for the Maxwell equations.

In general, we can regard the condition $\mathscr{A}_2 \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ as a closure relation, but also as an output feedback, as we will discuss in the next subsection.

[1] Actually in [42] it is shown that $\mathscr{T}$ is m-dissipative which in our situation is equivalent to being maximally dissipative, see Lemma 40.

4.4 Output feedback and systems

In this section we study output feedback, i.e., we look at $\mathscr{A}_{cl} = \mathscr{A}_0 - \mathscr{B}\mathscr{K}\mathscr{C}$. We can regard $z = \mathscr{A}_{cl}x_1$ as the solution of

$$z = \mathscr{A}_0 x_1 + \mathscr{B}u \text{ with } \mathscr{C}x_1 + \mathscr{K}^{-1}u = 0, \tag{72}$$

but then we would have to assume that $\mathscr{K}$ is invertible. In the following example we will show that this assumption can be removed.

Example 32 *It is easy to see that if $\mathscr{A}_0$ generates a contraction semigroup, so will $\mathscr{A}_0 - \mathscr{R}$ for any bounded, $\mathscr{R}$ with $-\mathscr{R}$ dissipative. In this example we show this using Theorem 25. For this we define*

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 & \mathscr{B}_0 & 0 \\ -\mathscr{B}_0^* & 0 & I \\ 0 & -I & -K \end{bmatrix},$$

where $(\mathscr{A}_0, D(\mathscr{A}_0))$ generates a contraction semigroup on the Hilbert space $\mathbb{Z}$, $\mathscr{B}_0 \in \mathscr{L}(\mathbb{U},\mathbb{Z})$, and $\mathscr{K} \in \mathscr{L}(\mathbb{U},\mathbb{U})$, with $\mathscr{K} + \mathscr{K}^* \geq 0$. Here $\mathbb{U}$ is another Hilbert space. The domain of $\mathscr{A}$ is given by $D(\mathscr{A}_0) \oplus \mathbb{U} \oplus \mathbb{U}$. Note that to apply our results we could even allow that $\mathscr{B}_0$ is unbounded, but here we apply it for bounded $\mathscr{B}_0$.

Next we choose $\mathbb{X}_1 = \mathbb{Z}$, $\mathbb{X}_2 = \mathbb{U} \oplus \mathbb{U}$, $\mathscr{E} = \mathscr{E}_I$ and $\mathscr{Q} = I$. Since $\mathscr{A}_0$ is a generator of a contraction semigroup, it is clear that the first three conditions of Assumptions 9 are fulfilled. It remains the show that $(\mathscr{E}_I, \mathscr{A})$ is regular.

Take s in the right half plane, then by the maximal dissipativity of $\mathscr{A}_0$,

$$(s\mathscr{E}_I - \mathscr{A}) \begin{bmatrix} x \\ u \\ y \end{bmatrix} = \begin{bmatrix} z \\ v \\ w \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ \mathscr{B}_0^* x - y \\ u + \mathscr{K}y \end{bmatrix} = \begin{bmatrix} (sI - \mathscr{A}_0)^{-1}z + (sI - \mathscr{A}_0)^{-1}\mathscr{B}_0 u \\ v \\ w \end{bmatrix}.$$

Substituting the expression for x into the second row, the following two equations remain to be solved:

$$\begin{bmatrix} \mathscr{B}_0^*(sI - \mathscr{A}_0)^{-1}\mathscr{B}_0 & -I \\ I & \mathscr{K} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} v - \mathscr{B}_0^*(sI - \mathscr{A}_0)^{-1}z \\ w \end{bmatrix}. \tag{73}$$

Since $\mathscr{B}_0$ is bounded and $\mathscr{A}_0$ generates a contraction semigroup, the transfer function $\mathscr{B}_0^*(sI - \mathscr{A}_0)^{-1}\mathscr{B}_0$ converges to zero as $s \to \infty$. Combined with the fact that $\left[\begin{smallmatrix} 0 & -I \\ I & \mathscr{K} \end{smallmatrix}\right]$ is boundedly invertible, we see that the left hand side of (73) is boundedly invertible for s sufficiently large.

So the conditions of Theorem 25 are satisfied, and we can construct the corresponding $\mathscr{A}_{red}$. It is given via

$$\mathscr{A}_{red} x_1 = \mathscr{A}_0 x_1 + \mathscr{B}_0 u \text{ with } -\mathscr{B}_0^* x_1 + y = 0 \text{ and } u + \mathscr{K}y = 0$$

The latter two properties give $u = -\mathscr{K}y = -\mathscr{K}\mathscr{B}_0^* x_1$, and so $\mathscr{A}_{red}$ becomes

$$\mathscr{A}_{red} = \mathscr{A}_0 - \mathscr{B}_0 \mathscr{K} \mathscr{B}_0^*,$$

which we can view as an output feedback on the system $\dot{x}_1(t) = \mathscr{A}_0 x_1(t) + \mathscr{B}_0 u(t), y(t) = \mathscr{B}_0^* x_1(t)$.

After applying the feedback we can again incorporate an input and an output, by considering the following $\mathscr{A}$ on the space $\mathbb{X} = \mathbb{Z} \oplus \mathbb{U}_1 \oplus \mathbb{U} \oplus \mathbb{U}$, where $\mathbb{U}_1$ is a Hilbert space, and $\mathscr{B}_1 \in \mathscr{L}(\mathbb{U}_1, \mathbb{Z})$.

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 & \mathscr{B}_1 & \mathscr{B}_0 & 0 \\ -\mathscr{B}_1^* & 0 & 0 & 0 \\ -\mathscr{B}_0^* & 0 & 0 & I \\ 0 & 0 & -I & \mathscr{K} \end{bmatrix}.$$

We split the space as $\mathbb{X}_1 = \mathbb{Z} \oplus \mathbb{U}_1$, $\mathbb{X}_2 = \mathbb{U} \oplus \mathbb{U}$, and so the last two rows of $\mathscr{A}$ form $\mathscr{A}_2$.

Choosing $\mathscr{E} = \mathscr{E}_I$ and $\mathscr{Q} = I$, then $\mathscr{A}$ obtained after applying the closure relation, is given by

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 - \mathscr{B}_0\mathscr{K}\mathscr{B}_0^* & \mathscr{B}_1 \\ -\mathscr{B}_1^* & 0 \end{bmatrix}$$

which is maximally dissipative. This implies that the system

$$\dot{z}(t) = (\mathscr{A}_0 - \mathscr{B}_0\mathscr{K}\mathscr{B}_0^*)z(t) + \mathscr{B}_1 u_1(t), \quad y_1(t) = -\mathscr{B}_1^* z(t)$$

is impedance passive. Note that with the choice of $\mathscr{Q} = \operatorname{diag}(\mathscr{Q}_1, I, I, I)$, we get impedance passivity with the storage function $q(z) = \langle z, \mathscr{Q}_1 z\rangle$, see [7, Theorem 7.5.4].

5 Existence of solutions on a subspace

In the previous section we have considered the operator $\mathscr{A}$ under the condition that $\mathscr{A}_2$ was injective on $\{0\} \oplus \mathbb{X}_2 \cap D(\mathscr{A})$. In the following theorem we use a stronger assumption and study the existence of solutions to (15).

Theorem 33 *Consider an adHDAE (15) with operators $\mathscr{E}$, $\mathscr{A}$, $\mathscr{Q}$ and the Hilbert spaces $\mathbb{X}, \mathbb{X}_1$, and $\mathbb{X}_2$ satisfying the conditions of Assumption 9. Define $\mathscr{W}_0 \subset \mathbb{X}_1$ as the first component of the kernel of $\mathscr{A}_2 \left[\begin{smallmatrix} \mathscr{Q}_1 \\ \mathscr{Q}_2 \end{smallmatrix}\right]$, i.e.,*

$$\mathscr{W}_0 = \{x_1 \in \mathbb{X}_1 \mid \exists\, x_2 \in \mathbb{X}_2 \text{ s.t. } \left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] \in D(\mathscr{A}) \text{ and } \mathscr{A}_2 \left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] = 0\}. \tag{74}$$

Let $\mathbb{X}_0 \subseteq \mathbb{X}_1$ be the closure of $\mathscr{W}_0$ in $\mathbb{X}_1$. If

$$\{y_1 \in \mathbb{X}_0 \mid \exists\, x_2 \in \mathbb{X}_2 \text{ s.t. } \left[\begin{smallmatrix} 0 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] \in D(\mathscr{A}) \text{ and } \mathscr{A} \left[\begin{smallmatrix} 0 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathscr{E}_1 y_1 \\ 0 \end{smallmatrix}\right]\} = \{0\}, \tag{75}$$

then the operator $\mathscr{A}_{red} : D(\mathscr{A}_{red}) \subset \mathbb{X}_0 \to \mathbb{X}_0$ generates a contraction semigroup on $\mathbb{X}_0$, where the domain $D(\mathscr{A}_{red})$ is defined as

$$\begin{aligned} D(\mathscr{A}_{red}) = \{x_1 \in \mathbb{X}_0 \mid \exists\, x_2 \in \mathbb{X}_2 \text{ such that } \left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] \in D(\mathscr{A}), \\ \mathscr{A}_2 \left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] = 0 \text{ and } \mathscr{E}_1^{-1}\mathscr{A}_1 \left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right] \in \mathbb{X}_0\} \end{aligned} \tag{76}$$

and for $x_1 \in D(\mathscr{A}_{red})$ the action of $\mathscr{A}_{red}$ is defined via

$$\mathscr{A}_{red} x_1 = \mathscr{E}_1^{-1}\mathscr{A}_1 \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{bmatrix}. \tag{77}$$

Proof. First we have to prove that $\mathscr{A}_{red}$ is well-defined. Note that $D(\mathscr{A}_{red}) \subset \mathscr{W}_0$. So if for a given $x_1 \in D(\mathscr{A}_{red})$ we have that x_2 and $\tilde{x}_2$ are such that the conditions on the domain are satisfied for $\left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 \tilde{x}_2 \end{smallmatrix}\right]$, then by the linearity of $\mathscr{A}_2$, we have that

$$\mathscr{A}_2 \begin{bmatrix} 0 \\ \mathscr{Q}_2 x_2 - \mathscr{Q}_2 \tilde{x}_2 \end{bmatrix} = 0.$$

Furthermore, we know that $y_1 := \mathscr{E}_1^{-1}\mathscr{A}_1 \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 x_2 \end{bmatrix}$ and $\tilde{y}_1 := \mathscr{E}_1^{-1}\mathscr{A}_1 \begin{bmatrix} \mathscr{Q}_1 x_1 \\ \mathscr{Q}_2 \tilde{x}_2 \end{bmatrix}$ are in $\mathbb{X}_0$. Since $\mathbb{X}_0$ is a linear space, we find that

$$y_1 - \tilde{y}_1 = \mathscr{E}_1^{-1}\mathscr{A}_1 \begin{bmatrix} 0 \\ \mathscr{Q}_2(x_2 - \tilde{x}_2) \end{bmatrix} \in \mathbb{X}_0.$$

Combining the two equations gives that

$$\mathscr{A}\begin{bmatrix}0\\ \mathscr{Q}_2(x_2-\tilde{x}_2)\end{bmatrix}=\begin{bmatrix}\mathscr{A}_1\\ \mathscr{A}_2\end{bmatrix}\begin{bmatrix}0\\ \mathscr{Q}_2(x_2-\tilde{x}_2)\end{bmatrix}=\begin{bmatrix}\mathscr{E}_1(y_1-\tilde{y}_1)\\ 0\end{bmatrix}$$

with $y_1-\tilde{y}_1\in\mathbb{X}_0$. Our assumption gives that $y_1=\tilde{y}_1$, and thus $\mathscr{A}_{red}x_1$ is unique, and so is well-defined.

We have

$$\begin{aligned}\langle\mathscr{A}_{red}x_1,x_1\rangle_{\mathscr{E}\mathscr{Q}}+\langle\mathscr{A}_{red}x_1,x_1\rangle_{\mathscr{E}\mathscr{Q}}&=\langle\mathscr{E}_1^{-1}\mathscr{A}_1\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix},\mathscr{E}_1^*\mathscr{Q}_1x_1\rangle+\langle\mathscr{E}_1^*\mathscr{Q}_1x_1,\mathscr{E}_1^{-1}\mathscr{A}_1\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}\rangle\\&=\langle\mathscr{A}\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix},\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}\rangle+\langle\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix},\mathscr{A}\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}\rangle\leq 0,\end{aligned}$$

where we have used that $\mathscr{A}_2\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}=0$. Hence $\mathscr{A}_{red}$ is dissipative.

Next we show that $sI-\mathscr{A}_{red}$ is onto, where s is a complex number with positive real part in the regularity assumption. For this, choose $y_1\in\mathbb{X}_0$. By the regularity assumption we know that there exists $\left[\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right]\in D(\mathscr{A}\mathscr{Q})$ such that

$$\begin{bmatrix}\mathscr{E}_1y_1\\ 0\end{bmatrix}=(s\mathscr{E}-\mathscr{A}\mathscr{Q})\begin{bmatrix}x_1\\ x_2\end{bmatrix}=s\begin{bmatrix}\mathscr{E}_1x_1\\ 0\end{bmatrix}-\mathscr{A}\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}. \tag{78}$$

The second row of this expression gives that

$$\mathscr{A}_2\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}=0$$

and so $x_1\in\mathscr{W}_0$. The first row of (78) gives

$$s\mathscr{E}_1x_1-\mathscr{A}_1\begin{bmatrix}\mathscr{Q}_1x_1\\ \mathscr{Q}_2x_2\end{bmatrix}=\mathscr{E}_1y_1.$$

Using that y_1 and x_1 are in $\mathbb{X}_0$, we get that $x_1\in D(\mathscr{A}_{red})$, and $(sI-\mathscr{A}_{red})x_1=y_1$. Hence $sI-\mathscr{A}_{red}$ is surjective for an $s\in\mathbb{C}^+$. By the Lumer-Phillips Theorem we conclude that $\mathscr{A}_{red}$ generates a contraction semigroup on $\mathbb{X}_0$. □

We note that if we have a classical solution of

$$\dot{x}_1(t)=\mathscr{A}x_1(t),$$

then $x_1(t)\in D(\mathscr{A})\subset\mathscr{W}_0\subset\mathbb{X}_0$ for all $t\geq 0$, and thus there exists an $x_2(t)$ such that

$$\mathscr{E}_1\dot{x}_1(t)=\mathscr{A}_1\begin{bmatrix}\mathscr{Q}_1x_1(t)\\ \mathscr{Q}_2x_2(t)\end{bmatrix}\text{ and }\mathscr{A}_2\begin{bmatrix}\mathscr{Q}_1x_1(t)\\ \mathscr{Q}_2x_2(t)\end{bmatrix}=0.$$

We can regard x_2 as the Lagrange multiplier enabling x_1 to stay in $\mathscr{W}_0$.

Example 34 *Consider the system (72) but with $\mathscr{K}=0$, i.e., let*

$$\mathscr{A}=\begin{bmatrix}\mathscr{A}_0 & \mathscr{B}_0\\ -\mathscr{B}_0^* & 0\end{bmatrix},$$

where $\mathscr{A}_0$ is maximally dissipative on the Hilbert space $\mathbb{X}_1$, and $\mathscr{B}_0\in\mathscr{L}(\mathbb{U},\mathbb{X}_1)$ where $\mathscr{B}_0$ is injective and has closed range, i.e., there exists $\beta>0$ such that $\|\mathscr{B}_0u\|\geq\beta\|u\|$, for all $u\in\mathbb{U}$. We choose $\mathbb{X}_2=\mathbb{U}$. To check the regularity for our class of $\mathscr{E}$ and $\mathscr{Q}$ it suffices to check it for $\mathscr{E}_I$ and $\mathscr{Q}=I$, see Corollary 15.

We first study the invertibility of the transfer function $G(s)=\mathscr{B}_0^*(sI-\mathscr{A})^{-1}\mathscr{B}_0$. It is well-known that $\lim_{s\to\infty}sG(s)=\mathscr{B}_0^*\mathscr{B}_0$, and by our assumption on $\mathscr{B}_0$ this inverse exists. So for s sufficiently large $sG(s)$

and thus also $G(s)$ *is boundedly invertible. Hence* $(\mathscr{E}_I, \mathscr{A})$ *is regular, and so is* $(\mathscr{E}, \mathscr{A}\mathscr{Q})$*. With this, we can define* $\mathbb{X}_0$ *and* $\mathscr{A}_{red}$*.*

Using equation (74) we get that $\mathscr{W}_0 = \{x_1 \in \mathbb{X}_1 \mid \mathscr{Q}_1 x_1 \in D(\mathscr{A}_0)$ *and* $\mathscr{Q}_1 x_1 \in \ker \mathscr{B}_0^*\}$*. Therefore,* $\mathbb{X}_0 = \mathscr{Q}_1^{-1}\overline{\ker \mathscr{B}_0^* \cap D(\mathscr{A}_0)}$*. In many cases the domain of* $\mathscr{A}_0$ *will be dense in the kernel of* $\mathscr{B}_0^*$*, and thus in that case* $\mathbb{X}_0 = \mathscr{Q}_1^{-1} \ker \mathscr{B}_0^*$*.*

The element y_1 *is in the set defined by equation (75) if* $\mathscr{B}_0 \mathscr{Q}_2 x_2 = \mathscr{E}_1 y_1$ *and* $\mathscr{Q}_1 y_1 \in \ker \mathscr{B}_0^*$*. Thus* $\mathscr{B}_0^* \mathscr{Q}_1 \mathscr{E}_1^{-1} \mathscr{B}_0 \mathscr{Q}_2 x_2 = 0$*, which implies that* $\langle \mathscr{B}_0 \mathscr{Q}_2 x_2, \mathscr{Q}_1 \mathscr{E}_1^{-1} \mathscr{B}_0 \mathscr{Q}_2 x_2 \rangle = 0$*. Since* $\mathscr{Q}_1 \mathscr{E}_1^{-1}$ *is coercive, this gives* $\mathscr{B}_0 \mathscr{Q}_2 x_2 = 0$ *and thus* $\mathscr{E}_1 y_1 = 0$*. The invertibility of* $\mathscr{E}_1$ *finally gives* $y_1 = 0$*.*

Thus, all the conditions of Theorem 33 are satisfied. We choose $\mathscr{E} = \mathscr{E}_I$ *and* $\mathscr{Q} = I$*, to study the* $\mathscr{A}$ *constructed in Theorem 33.*

$$\hat{\mathscr{A}} x_1 = \mathscr{A}_0 x_1 + \mathscr{B}_0 u, \text{ with } x_1 \in D(\mathscr{A}_0), \mathscr{B}_0^* x_1 = 0, \text{ and } \mathscr{B}_0^*(\mathscr{A}_0 x_1 + \mathscr{B}_0 u) = 0.$$

The last expression gives $u = -(\mathscr{B}_0^* \mathscr{B}_0)^{-1} \mathscr{B}_0^* \mathscr{A}_0 x_1$*, and so on* $\mathbb{X}_0$ *we have the operator*

$$\mathscr{A}_{red} x_1 = \left(\mathscr{A}_0 - \mathscr{B}_0 (\mathscr{B}_0^* \mathscr{B}_0)^{-1} \mathscr{B}_0^* \mathscr{A}_0\right) x_1.$$

Theorem 33 states that there is a well-defined dynamics on this space. If we interpret the second state component as the output, then this $\mathbb{X}_0$ *has the interpretation as the output nulling subspace. It is well-known that the largest output nulling subspace exists when* $\mathscr{B}_0^* \mathscr{B}_0$ *is invertible, see [8] or [49].*

In general, when $\mathscr{C} \in \mathscr{L}(\mathbb{X}_1, \mathbb{U})$ *is such that there exists a coercive* $\mathscr{Q}_1 \in \mathscr{L}(\mathbb{X}_1)$ *such that* $\mathscr{C} = \mathscr{B}_0^* \mathscr{Q}_1$*, then we get, with* $\mathscr{E} = \mathscr{E}_I$ *and* $\mathscr{Q} = \operatorname{diag}(\mathscr{Q}_1, I)$*, that* $\mathbb{X}_0 = \ker \mathscr{C}$*, and*

$$\hat{\mathscr{A}} x_1 = \left(\mathscr{A}_0 - \mathscr{B}_0 (\mathscr{C} \mathscr{B}_0)^{-1} \mathscr{C} \mathscr{A}_0\right) \mathscr{Q}_1 x_1.$$

In the following example we study the class studied in Theorem 22. However, the applications of this class are different, it contains e.g. the Oseen or Stokes equation, see [12] and [36]. The setup is similar as for the impedance passive systems studied in Subsection 4.3.

Example 35 *Let* $\mathbb{V}$ *be a real Hilbert space such that* $\mathbb{V} \hookrightarrow \mathbb{X}_1 = \mathbb{X}_1^* \hookrightarrow \mathbb{V}^*$*, Let* $\mathscr{A}_0 \in \mathscr{L}(\mathbb{V}, \mathbb{V}^*)$*,* $\mathscr{B}_0 \in \mathscr{L}(\mathbb{U}, \mathbb{V}^*)$*, where* $\mathbb{U}$ *is a second (real) Hilbert space. So* $\mathscr{B}_0^* \in \mathscr{L}(\mathbb{V}, \mathbb{U}^*)$*. We identify* $\mathbb{U}^*$ *with* $\mathbb{U}$*. We assume that* $\mathscr{A}_0$ *is dissipative and* $\mathscr{B}_0$ *is injective and has closed range.*

With these operators we define, see also (29) and (30),

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 & \mathscr{B}_0 \\ -\mathscr{B}_0^* & 0 \end{bmatrix} \tag{79}$$

with domain

$$D(\mathscr{A}) = \{ \begin{bmatrix} v \\ u \end{bmatrix} \in \mathbb{V} \oplus \mathbb{U} \mid \mathscr{A}_0 v + \mathscr{B}_0 u \in \mathbb{X}_1 \}.$$

By Theorem 22 we know that $(\mathscr{E}_I, \mathscr{A})$ *is regular. So we can apply Theorem 33 on this class.*

By the definition of $\mathscr{A}$ *we have that*

$$\mathscr{W}_0 = \{x_1 \in \mathbb{V} \mid \exists\, x_2 \in \mathbb{U} \text{ s.t. } \mathscr{A}_0 x_1 + \mathscr{B}_0 x_2 \in \mathbb{X}_1, \text{ and } \mathscr{B}_0^* x_1 = 0\}.$$

Next we study the solution set of equation (75). Let $y_1 \in \mathbb{X}_0 = \overline{\mathscr{W}_0}$*, i.e., the closure of* $\mathscr{W}_0$ *in* $\mathbb{X}_1$*, be such that there exists an* $u \in \mathscr{U}$ *is such that*

$$\mathscr{A} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}. \tag{80}$$

From the definition of the domain of $\mathscr{A}$ *we obtain that* $\mathscr{B}_0 u \in \mathbb{X}_1$*. If* $\mathscr{B}_0$ *is completely unbounded, then this would imply that* $u = 0$*, and thus* $y_1 = 0$*. Otherwise, since* $y_1 \in \mathbb{X}_0$ *there exists a sequence* $z_n \in \mathscr{W}_0 \subset \mathbb{V}$

such that $z_n \to y_1$ in $\mathbb{X}_1$. In particular, $\mathscr{B}_0^* z_n = 0$. Combining this with the fact that $y_1 = \mathscr{B}_0 u$, see (80), we find

$$\langle y_1, y_1 \rangle_{\mathbb{X}_1} = \lim_{n\to\infty} \langle z_n, \mathscr{B}_0 u \rangle_{\mathbb{X}_1} = \lim_{n\to\infty} \langle z_n, \mathscr{B}_0 u \rangle_{\mathbb{V},\mathbb{V}^*} = \lim_{n\to\infty} \langle \mathscr{B}_0^* z_n, u \rangle_{\mathbb{U}} = 0.$$

Hence $y_1 = 0$. Thus the conditions of Theorem 33 are satisfied.

A concrete application of the set-up in the previous example is given next.

Example 36 *Consider, as in [12] a linearized Navier-Stokes equation and given by*

$$\begin{aligned} \frac{\partial v}{\partial t} - \alpha \Delta v + \nabla p &= 0 \\ \nabla^T v &= 0, \end{aligned}$$

on a spatial domain Ω.

For the abstract set-up of Example 35 we choose $\mathbb{V} = H_0^1(\Omega)$, $\mathbb{X}_1 = L^2(\Omega)$, and $\mathbb{U} = \mathbb{X}_2 = L^2(\Omega)/\mathbb{R}$, i.e., two functions in $\mathbb{U}$ are considered to be the same if they differ by a constant. Furthermore, $\mathscr{A}$ is taken as

$$\mathscr{A} = \begin{bmatrix} \mathscr{A}_0 & \mathscr{B}_0 \\ -\mathscr{B}_0^* & 0 \end{bmatrix} = \begin{bmatrix} \alpha\Delta & -\nabla \\ \nabla^T & 0 \end{bmatrix}.$$

Since for $v, w \in \mathbb{V}$

$$\int_\Omega (\Delta v)\, w \, d\omega = -\int_\Omega \nabla v \cdot \nabla w \, d\omega,$$

we see that $\mathscr{A}_0$ is dissipative. Furthermore, it satisfies the Gårding inequality, see [12] for the proof in the more general case of the linearized Navier-Stokes and Oseen equation. Furthermore, $\mathscr{B}_0$ is injective, has closed range and satisfies the condition (33), see e.g. [6]. Hence if we choose

$$\mathscr{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{Q} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

then it fits the framework of Example 35. Note that $\mathbb{X}_0$ is now the space of divergence free functions.

6 Conclusion and possible extensions

Abstract linear dissipative Hamiltonian differential-algebraic equations (DAEs) on Hilbert spaces are studied. A characterization is given when these are associated with singular and regular operator pairs. It is shown that due to closure relations and structural properties this class of operator equations arises typically when studying classical evolution equations. This is illustrated by several applications.

However, this class does not only arises when the state spaces are Hilbert spaces, and these abstract DAEs are not restricted to linear systems. To extend the presented theory for dissipative systems on a Banach space, the article [40] can serve as a starting point. Among others it is shown there that Example 28 can be treated in the context of Banach spaces as well.

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7 Appendix on dissipative operators

Dissipative operators are important in this paper, and so we list some of their properties. We begin with its definition.

Definition 37 *Let $\mathbb{X}$ be a (complex) Hilbert space. Then $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ is* dissipative *if*

$$\mathrm{Re}\langle \mathscr{A}x, x\rangle \leq 0 \quad \textit{for all } x \in D(\mathscr{A}). \tag{81}$$

The following equivalent characterization is very useful. For a proof we refer to e.g. Proposition 6.1.5 of [23].

Lemma 38 *The operator $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \mapsto \mathbb{X}$ is dissipative if and only if*

$$\|(\lambda I - \mathscr{A})x\| \geq \lambda \|x\|, \quad \textit{for all } x \in D(\mathscr{A}), \lambda > 0. \tag{82}$$

For complex s with positive real part, it is easy to see that we have to replace (82) by

$$\|(sI - \mathscr{A})x\| \geq \mathrm{Re}(s)\|x\|.$$

From this we see immediately that a dissipative $\mathscr{A}$ will not have eigenvalues in $\mathbb{C}^+$. Furthermore, when $(sI - \mathscr{A})$ is surjective, this inequality implies that $(sI - \mathscr{A})$ is boundedly invertible. Secondly, (82) implies that $sI - \mathscr{A}$ is *closable*, and thus $\mathscr{A}$ is. This means that there exists an extension of $\mathscr{A}$ which we denote by $\overline{\mathscr{A}}$ such if $x_n \in D(\mathscr{A})$ converged to x and $\mathscr{A}x_n$ converge to y, then $x \in D(\overline{\mathscr{A}})$ and $\mathscr{A}x = y$. Furthermore, this closure is dissipative, see e.g. [1].

Based on this consider the following two concepts.

Definition 39 *Let $\mathbb{X}$ be a Hilbert space, and $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \mapsto \mathbb{X}$ a dissipative operator.*

1. *$\mathscr{A}$ is* m-dissipative *if the range of $\lambda I - \mathscr{A} = \mathbb{X}$ for a $\lambda > 0$;*
2. *$\mathscr{A}$ is* maximally dissipative *if there does not exists an extension of $\mathscr{A}$ which is also dissipative.*

Lemma 40 *Let $\mathbb{X}$ be a Hilbert space, and $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \mapsto \mathbb{X}$ a dissipative operator. Then it is m-dissipative if and only if it is maximally dissipative.*

For the proof we refer to Corollary 2.27 of [31]. Using this lemma we do not distinguish the two concepts, and we have chosen to use the term maximally dissipative when 1. or 2. holds, see Definition 39.

The importance of dissipative operators is clear from the *Lumer-Phillips Theorem*.

Theorem 41 *Let $\mathbb{X}$ be a Hilbert space, and $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \mapsto \mathbb{X}$ a linear operator. Then the following are equivalent:*

1. *$\mathscr{A}$ is maximally dissipative;*
2. *$\mathscr{A}$ is the infinitesimal generator of a contraction semigroup on $\mathbb{X}$;*
3. *$\mathscr{A}$ is closed and densely defined, and $\mathscr{A}$ and $\mathscr{A}^*$ are dissipative.*

For the proof of (1) $\Leftrightarrow$ (2) we refer to [23, Theorem 6.1.7], and for (2) $\Leftrightarrow$ (3) to [7, Corollary 2.3.3].

We end this appendix with a lemma.

Lemma 42 *If $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \mapsto \mathbb{X}$ a dissipative operator which is boundedly invertible, then it is maximally dissipative.*

Proof. The proof follows from the fact that the resolvent set of an operator is always open. Thus there exists a $\lambda > 0$ such that $\lambda I - \mathscr{A}$ is boundedly invertible, and in particular its range equals $\mathbb{X}$. □

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