DAE Panel

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Index Concepts for Linear Differential-Algebraic Equations in Infinite Dimensions

MEHMET ERBAY, BIRGIT JACOB, KIRSTEN MORRIS, TIMO REIS, AND CAREN TISCHENDORF

Abstract: Different index concepts for regular linear differential-algebraic equations are defined and compared in the general Banach space setting. For regular finite dimensional linear differential-algebraic equations, all these indices exist and are equivalent. For infinite dimensional systems, the situation is more complex. It is proven that although some indices imply others, in general they are not equivalent. The situation is illustrated with a number of examples.

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1 Introduction

In this article we look at various index terms for infinite dimensional differential-algebraic systems (DAE) of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + f(t), \quad t \geq 0, \tag{1}$$

where $E\colon X \to Z$ is a bounded linear operator (denoted by $E \in \mathscr{L}(X,Z)$), $(A, \mathrm{dom}(A))$ is a closed and densely defined linear operator from X to Z and $f\colon [0,\infty) \to Z$. Throughout this article, X and Z are complex Banach spaces and the DAE (1) is assumed to be regular. That is,

$$\rho(E,A) := \left\{ \lambda \in \mathbb{C} \;\middle|\; (\lambda E - A)^{-1} \in \mathscr{L}(Z,X) \right\} \neq \emptyset.$$

By a *solution* of (1) we mean a *classical solution*, that is, a function $x\colon [0,\infty) \to \mathrm{dom}(A)$ such that $Ex(\cdot)$ is continuously differentiable as a function with values in Z, and (1) is satisfied for every $t \geq 0$.

The index of a DAE can be defined in a number of various ways. Examples include the differentiation index [6, 7, 13, 17, 19], the perturbation index [7, 22], the strangeness index [7, 22], the nilpotency index [17], the geometric index [22, 17, 27, 26], the resolvent index [10, 11, 32, 33] and the radiality index [15, 31]. Not all of these indices are defined in the infinite dimensional case. For instance, the nilpotency index of a DAE demands a Weierstraß form (defined formally below), which is not always available.

Our aim in writing this paper is to “collect” all the index terms that are applicable in the infinite dimensional case, and to characterize and compare them to each other. In particular, we investigate the resolvent index $p_{\text{res}}^{(E,A)}$, the chain index $p_{\text{chain}}^{(E,A)}$, the radiality index $p_{\text{rad}}^{(E,A)}$, the nilpotency index $p_{\text{nilp}}^{(E,A)}$, the differentiation index $p_{\text{diff}}^{(E,A)}$ and the perturbation index $p_{\text{pert}}^{(E,A)}$. Several of these indices have not previously been defined for infinite dimensional systems.

One of our main results is that if all the indices mentioned in the previous paragraph exist then

$$p_{\text{rad}}^{(E,A)} + 1 \geq p_{\text{res}}^{(E,A)} \geq p_{\text{nilp}}^{(E,A)} = p_{\text{chain}}^{(E,A)}.$$

If in addition $A|_{X^1}$ generates a C_0-semigroup, where X^1 denotes the biggest subspace of X on which E is invertible, then

$$p_{\text{rad}}^{(E,A)} + 1 \geq p_{\text{res}}^{(E,A)} \geq p_{\text{nilp}}^{(E,A)} = p_{\text{diff}}^{(E,A)} = p_{\text{chain}}^{(E,A)} = p_{\text{pert}}^{(E,A)}.$$

Note, that the generation of a C_0-semigroup guarantees the existence and uniqueness of mild solutions of the corresponding Cauchy problem, see [8] for further information. Furthermore, Proposition 6.9 implies that in the finite dimensional case, equality holds in all these bounds.

2 Weierstraß form

Consider a differential-algebraic system of the form (1), denoted by (E,A). Throughout the article we will assume that (E,A) is regular, that is

$$\rho(E,A) := \{ \lambda \in \mathbb{C} \mid (\lambda E - A)^{-1} \in \mathscr{L}(X,Z) \} \neq \emptyset.$$

Let $\tilde{X},\tilde{Z}$ be Banach spaces, $\tilde{E} \in \mathscr{L}(\tilde{X},\tilde{Z})$ and $\tilde{A}\colon \operatorname{dom}(\tilde{A}) \subseteq \tilde{X} \to \tilde{Z}$ closed and densely defined.

Definition 2.1. *Two differential-algebraic systems $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$ and $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x = \tilde{A}x$ are equivalent, denoted by $(E,A) \sim (\tilde{E},\tilde{A})$, if there are two bounded isomorphisms $P\colon X \to \tilde{X}$, $Q\colon Z \to \tilde{Z}$, such that $E = Q^{-1}\tilde{E}P$ and $A = Q^{-1}\tilde{A}P$.*

Remark 2.2. *The equivalence of two differential-algebraic systems is an equivalence relation. This follows from the fact that the concatenation of two bounded isomorphisms is a bounded isomorphism.*

Definition 2.3. *A bounded operator $N \in \mathscr{L}(X)$ is* nilpotent*, if there exist a $p \in \mathbb{N}$, such that $N^l \neq 0$ for all $l < p$ and $N^p = 0$. The integer p is called the* degree of nilpotency.

This definition may be slightly different in other references. For example, in [31] the degree of nilpotency is $p-1$ and not p.

Definition 2.4. *The DAE (1) has a Weierstraß form, if there exist a Banach space $Y = Y^1 \oplus Y^2$, such that*

$$(E,A) \sim \left(\begin{bmatrix} I_{Y^1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{Y^2} \end{bmatrix} \right), \tag{2}$$

where $N\colon Y^2 \to Y^2$ is a bounded linear nilpotent operator, $A_1\colon \operatorname{dom}(A_1) \subseteq Y^1 \to Y^1$ is a linear operator and I_{Y^i} indicates the identity operator on the associated subspace Y^i, $i = 1,2$.

This form is also known variously as the *quasi-Weierstraß form* [2] or *Weierstraß canonical form* [17]. In finite dimensions with complex arithmetic the operator A_1 can be chosen to be a matrix in Jordan canonical form J. In this case, the Weierstraß form, and hence the degree of nilpotency of N, always exist. Furthermore the Weierstraß form is unique up to isomorphisms and therefore the nilpotency degree of N is uniquely determined. To be more precise, assume that (E,A) has two different Weierstraß forms $\left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_i \end{smallmatrix}\right], \left[\begin{smallmatrix} J_i & 0 \\ 0 & I \end{smallmatrix}\right]\right)$, $i = 1,2$. Then the sizes of the matrices in Jordan canonical form J_1, J_2 and of the nilpotent operators N_1, N_2 coincide, as well as the degree of nilpotency of these nilpotent operators [17, Lem. 2.10]. Certainly, this only holds for the finite dimensional case and under the assumption, that (E,A) is regular (as assumed in the introduction). Further studies regarding the regularity can be found in [20, 23, 21].

In the next sections we will define a variety of different index terms and generalise/adapt them for the infinite dimensional case. Most of the terms are already known in the finite dimensional case.

3 Resolvent index

The *resolvent index* has already been defined in [10, p. 5], [32, p. 8] and [11, ch. 6.1]. It has the advantage that it does not require a Weierstraß form. Thus, this definition can be easily extended to the infinite dimensional case. The only difficulty encountered in calculating this index is the calculation of the resolvent and its growth rate, which is a greater hurdle in the infinite dimensional case. In this context, we call $(\lambda E - A)^{-1}$ the *resolvent of* (E,A) for $\lambda \in \rho(E,A)$.

Definition 3.1 (resolvent index).
The resolvent index of (E,A) is the smallest integer $p_{\text{res}}^{(E,A)} \in \mathbb{N}_0$, such that there exist a $\omega \in \mathbb{R}$, $C > 0$ with $(\omega,\infty) \subseteq \rho(E,A)$ and

$$\left\|(\lambda E - A)^{-1}\right\| \leq C|\lambda|^{p_{\text{res}}^{(E,A)}-1} \tag{3}$$

for all $\lambda \in (\omega,\infty)$. The resolvent index is called a complex resolvent index, denoted by $p_{\text{c,res}}^{(E,A)} \in \mathbb{N}_0$, if $\mathbb{C}_{\text{Re}>\omega} := \{ \lambda \in \mathbb{C} \mid \text{Re}\,\lambda > \omega \} \subseteq \rho(E,A)$ and (3) holds for $p_{\text{c,res}}^{(E,A)}$.

Note that the resolvent index can also defined in a weaker form as seen in [11, ch. 5& 6]. Clearly, $p_{\text{res}}^{(E,A)} \leq p_{\text{c,res}}^{(E,A)}$. The next proposition shows that this index is uniquely defined.

Proposition 3.2. *The resolvent index, given that it exists, is uniquely defined. To be more precise, let $(E,A) \sim (\tilde{E},\tilde{A})$. Then $p_{\text{res}}^{(E,A)} = p_{\text{res}}^{(\tilde{E},\tilde{A})}$ and $p_{\text{c,res}}^{(E,A)} = p_{\text{c,res}}^{(\tilde{E},\tilde{A})}$.*

Proof. Since $(E,A) \sim (\tilde{E},\tilde{A})$ there exist two isomorphisms $P\colon X \to \tilde{X}$, $Q\colon Z \to \tilde{Z}$ with

$$E = Q^{-1}\tilde{E}P \quad \text{and} \quad A = Q^{-1}\tilde{A}P.$$

Assume that (E,A) has resolvent index $p_{\text{res}}^{(E,A)}$. Then, there exist a $C > 0$, $\omega \in \mathbb{R}$, such that $(\omega,\infty) \subseteq \rho(E,A)$ and

$$\|(\lambda E - A)^{-1}\| \leq C|\lambda|^{p_{\text{res}}^{(E,A)}-1}, \quad \lambda > \omega.$$

Since P and Q are bounded isomorphisms and

$$(\lambda\tilde{E} - \tilde{A})^{-1} = P(\lambda E - A)^{-1}Q^{-1}, \quad \lambda > \omega$$

it follows that $(\omega,\infty) \subseteq \rho(E,A) \subseteq \rho(\tilde{E},\tilde{A})$ and

$$\|(\lambda\tilde{E} - \tilde{A})^{-1}\| \leq \|P\| \left\|(\lambda E - A)^{-1}\right\| \left\|Q^{-1}\right\| \leq \tilde{C}|\lambda|^{p_{\text{res}}^{(E,A)}-1}, \quad \lambda > \omega,$$

for $\tilde{C} := C\|P\| \left\|Q^{-1}\right\|$. Thus, the resolvent index of $(\tilde{E},\tilde{A})$ is at most $p_{\text{res}}^{(E,A)}$. The other estimate follows from an equivalent argument and switching (E,A) and $(\tilde{E},\tilde{A})$. The statement concerning the complex resolvent index follows similarly. □

Next, we will show the existence of the (complex) resolvent index for a special class of systems, namely $X = Z$ a Hilbert space, E is non-negative and A is dissipative. Note that we call E non-negative, denoted by $E \geq 0$, if $\langle Ex,x\rangle \geq 0$ for all $x \in X$ and we call A dissipative, if $\text{Re}\langle Ax,x\rangle \leq 0$ for all $x \in \text{dom}(A)$.

These conditions are met by *port-Hamiltonian DAEs* or *abstract dissipative DAEs*, among others (see [11, Sec. 7] and [23]).

Theorem 3.3. *Let $X = Z$ be a Hilbert space, $E \in \mathscr{L}(X)$ be non-negative self-adjoint and $A\colon \operatorname{dom}(A) \subseteq X \to X$ be dissipative. If there exist a $\omega > 0$, such that $(\omega,\infty) \subseteq \rho(E,A)$, then $p_{\text{res}}^{(E,A)} \le 2$. If also $\mathbb{C}_{\text{Re}>\omega} \subseteq \rho(E,A)$, then $p_{\text{c,res}}^{(E,A)} \le 3$.*

Proof. For any $x \in X\setminus\{0\}$ and $\lambda \in \rho(E,A)\cap\mathbb{C}_{\text{Re}>0}$ define $z = (\lambda E - A)^{-1}x$. By using the dissipativity of A and the non-negativity of E we deduce

$$\operatorname{Re}\langle(\lambda E-A)^{-1}x,x\rangle = \operatorname{Re}\langle z,(\lambda E-A)z\rangle = \langle z,Ez\rangle \operatorname{Re}\lambda - \operatorname{Re}\langle z,Az\rangle \ge 0$$

for all $\lambda \in \rho(E,A)\cap\mathbb{C}_{\text{Re}>0}$. Thus, for every $x \in X$ the function

$$f_x\colon \lambda \mapsto \langle(\lambda E-A)^{-1}x,x\rangle + \|x\|^2$$

is positive real on $\mathbb{C}_{\text{Re}>\omega}$, i.e. $\operatorname{Re} f_x(\lambda) > 0$ for all $\lambda \in \mathbb{C}_{\mathbb{R}>\omega}$. Let $\mu > \omega$. Using [12, Thm. 3] we obtain

$$|f_x(\lambda)| \le |f_x(\mu)|\,\frac{|\lambda|^2+|\mu|^2+3|\lambda\mu|}{\mu\operatorname{Re}\lambda} \le |f_x(\mu)|\,\frac{5}{\mu}\frac{|\lambda|^2}{\operatorname{Re}\lambda},$$

for all $\lambda \in \mathbb{C}_{\text{Re}>\mu}\cap\rho(E,A)$. Hence, together with the Riesz representation theorem we derive

$$\left\|(\lambda E-A)^{-1}\right\| = \sup_{\substack{x,y\in X\\ \|x\|=\|y\|=1}} \left|\langle(\lambda E-A)^{-1}x,y\rangle\right| \le \sup_{\substack{x\in X\\ \|x\|=1}} 2|f_x(\mu)|\,\frac{5}{\mu}\frac{|\lambda|^2}{\operatorname{Re}\lambda} \le K\frac{|\lambda|^2}{\operatorname{Re}\lambda},$$

for all $\lambda \in \mathbb{C}_{\text{Re}>\omega}$ with $K = (\left\|(\mu E-A)^{-1}\right\|+1)\frac{10}{\mu}$. Thus, the complex resolvent index is at most 3 and if $\lambda > \omega$ is real, then $\frac{|\lambda|^2}{\operatorname{Re}\lambda} = \lambda$ and the resolvent index is at most 2. □

Example 3.4. *($p_{\text{res}}^{(E,A)} = 2$ and $p_{\text{c,res}}^{(E,A)} = 3$.) We formally define the infinite block diagonal matrices $A_o = \operatorname{diag}(A_0,A_1,A_2,\dots)$, $E_o = \operatorname{diag}(E_0,E_1,E_2,\dots)$ with*

$$A_0 = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad A_k = \begin{bmatrix}0 & \sqrt{k^4+1}\\ -\sqrt{k^4+1} & -2\end{bmatrix},$$

$$E_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad E_k = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad k \in \mathbb{N}.$$

and the infinite matrix with one column

$$B_o = \begin{bmatrix}B_0\\ B_1\\ \vdots\end{bmatrix},$$

whereby

$$B_0 = \begin{bmatrix}0\\ 1\end{bmatrix}, \qquad B_k = \begin{bmatrix}0\\ k^{\frac{5}{4}}\end{bmatrix}, \quad k \in \mathbb{N}.$$

Consider the space

$$\mathscr{D} = \left\{ \left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right) \in \ell^2\times\mathbb{C} \;\middle|\; A_o x_1 + B_o x_2 \in \ell^2 \right\},$$

which is a Hilbert space endowed with the norm

$$\|\left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right)\|_{\mathscr{D}}^2 = \|x_1\|_{\ell^2}^2 + |x_2|^2 + \|A_o x_1 + B_o x_2\|_{\ell^2}^2.$$

For $\binom{x_1}{x_2} \in \mathscr{D}$*, by grouping successive sequence elements, we express* x_1 *as a sequence of vectors in* $\mathbb{C}^2$*, i.e.,*

$$x_1 = (x_{1k}), \quad x_{1k} \in \mathbb{C}^2. \tag{4}$$

Then $z = (z_k) := A_o x_1 + B_o x_2 \in \ell^2$ *implies that*

$$x_{1k} = A_k^{-1} z_k - A_k^{-1} B_k, \quad k \in \mathbb{N}. \tag{5}$$

Since,

$$(sI_2 - A_k)^{-1} = \frac{1}{s^2 + 2s + k^4 + 1} \begin{bmatrix} s+2 & \sqrt{k^4+1} \\ -\sqrt{k^4+1} & s \end{bmatrix}, \quad k \in \mathbb{N},\ s \in \mathbb{C}_{\mathrm{Re}\geq 0}, \tag{6}$$

we obtain that (for any norm in $\mathbb{C}^{2\times 2}$*), for all* $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$*, there exists some* $c(s) > 0$ *with*

$$\|(sI_2 - A_k)^{-1}\| \leq \frac{c(s)}{k^2}, \quad k \in \mathbb{N}. \tag{7}$$

Combining (7) *with* $e_2^\top A_k^{-1} B_k = 0$ *(where* $e_2 \in \mathbb{R}^2$ *is the second canonical unit vector);* (5) *then yields that*

$$\|\tbinom{x_1}{x_2}\|_{\mathscr{D}} \geq \|(z_k)\|_{\ell^2} \geq c(0)\,\|(k^2 e_2^\top x_k)\|_{\ell^2}.$$

In particular, the formal adjoint

$$C_o := B_o^\top = (B_0^\top, B_1^\top, \ldots)$$

defines a bounded linear operator from $\mathscr{D}$ *to* $\mathbb{C}$ *via*

$$\tbinom{x_1}{x_2} \mapsto C_o x_1.$$

For later use, we record that (7) *further implies that*

$$\left((sI_2 - A_k)^{-1} B_k\right)_{k\in\mathbb{N}} \in \ell^2. \tag{8}$$

Now consider the operators

$$E := \begin{bmatrix} E_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_o & B_o & 0 \\ -C_o & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

defined by (infinite) block matrices introduced above. Then, for $X = \ell^2 \times \mathbb{R} \times \mathbb{R}$*, we have that* E *is bounded, self-adjoint and non-negative. Further, by completeness of* $\mathscr{D}$ *and the above shown boundedness property of* C_o*, we have that* $A : \mathrm{dom}(A) \to X$ *with* $\mathrm{dom}(A) = \mathscr{D} \times \mathbb{C}$ *is closed. Since* C_o *is the formal adjoint of* B_o*, and the matrix* A_k *is dissipative for all* $k \in \mathbb{N}$*,* A *is a dissipative operator.*

Next we analyse the resolvent of (E, A)*. First, by invertibilty of* $(sE_0 - A_0)$ *for all* $s \in \mathbb{C}$*, we obtain from* (7) *that* $sE_o - A_o$ *has a bounded inverse for all* $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$*, with*

$$(sE_o - A_o)^{-1} = \mathrm{diag}((sE_0 - A_0)^{-1}, (sI_2 - A_1)^{-1}, (sI_2 - A_2)^{-1}, (sI_2 - A_3)^{-1}, \ldots), \quad s \in \mathbb{C}_{\mathrm{Re}\geq 0}.$$

Moreover, (7) *implies that for all* $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$*,* $(sE_o - A_o)^{-1} B_o$ *and* $C_o(sE_o - A_o)^{-1}$ *define bounded operators from* $\mathbb{C}$ *to* ℓ^2 *and from* ℓ^2 *to* $\mathbb{C}$*, respectively.*

We now show that $C_o(sE_o - A_o)^{-1} B_o$ *is well-defined for all* $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$*: First, since*

$$A_o(sE_o - A_o)^{-1} B_o + B_o = (sE_o - (sE_o - A_o)(sE_o - A_o)^{-1} B_o + B_o = sE_o(sE_o - A_o)^{-1} B_o \in \ell^2,$$

it follows that

$$\begin{pmatrix} (sE_o - A_o)^{-1} B_o \\ 1 \end{pmatrix} \in \mathscr{D}.$$

Then the previously proven property of C_o yields that $C_o(sE_o - A_o)^{-1}B_o \in \mathbb{C}$ is well-defined.
So far, we have proven that the operators $(sE_o - A_o)^{-1} : \ell^2 \to \ell^2$, $(sE_o - A_o)^{-1}B_o : \mathbb{C} \to \ell^2$, $C_o(sE_o - A_o)^{-1} : \ell^2 \to \mathbb{C}$ are bounded and well-defined for all $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$. Combining this with $C_o(sE_o - A_o)^{-1}B_o \in \mathbb{C}$, these results imply that

$$H(s) = \begin{bmatrix} (sE_o - A_o)^{-1} & 0 & (sE_o - A_o)^{-1}B_o \\ 0 & 0 & 1 \\ C_o(sE_o - A_o)^{-1} & -1 & C_o(sE_o - A_o)^{-1}B \end{bmatrix}.$$

is a bounded linear operator from $X = \ell^2 \times \mathbb{C} \times \mathbb{C}$ to $\mathscr{X}$ for all $s \in \mathbb{C}_{\mathrm{Re}\geq 0}$. A straightforward calculation moreover shows that $H(s)(sE - A)x = x$ and $(sE - A)H(s)z = z$ for all $x \in X$ and $z \in \mathrm{dom}(A)$, which implies that $H(s) = (sE - A)^{-1}$. Thus, (E, A) satisfies the conditions of Theorem 3.3.

It will now be shown that $p^{(E,A)}_{\mathrm{res}} = 2$ and $p^{(E,A)}_{\mathrm{c,res}} = 3$. First we note that, by using the diagonal structure of E_o and A_o,

$$C_o(sE_o - A_o)^{-1}B_o = C_0(sE_0 - A_0)^{-1}B_0 + \sum_{k=1}^{\infty} C_k(sI_2 - A_k)^{-1}B_k.$$

Since $C_0(sE_0 - A_0)^{-1} = s$ and, by (6),

$$C_k(sI_2 - A_k)^{-1}B_k = \frac{k^{\frac{5}{2}}s}{s^2 + 2s + k^4 + 1},$$

it follows that

$$C_o(sE_o - A_o)^{-1}B_o = s + \sum_{k=1}^{\infty} \frac{k^{\frac{5}{2}}s}{s^2 + 2s + k^4 + 1}.$$

of (E, A). Since $C_o(sE_o - A_o)^{-1}B_o \geq s$ for real and non-negative s, the resolvent $H(s)(sE - A)^{-1}$ grows at least linearly along the real axis and together with Theorem 3.3 this implies $p^{(E,A)}_{\mathrm{res}} = 2$.[1]

Now we consider growth in the entire right-half-plane. Letting $\sigma > 0$ and $s_n := \sigma + in^2$ for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{Re}\left(C_o(s_nE_o - A_o)^{-1}B_o\right) &= \sigma + \sum_{k=1}^{\infty} \frac{k^{\frac{5}{2}}(\sigma((1+\sigma)^2 + k^4) + (2+\sigma)n^4)}{((1+\sigma)^2 + (k^4 - n^4))^2 + 4(1+\sigma)^2 n^4} \\
&\geq \sigma + \frac{n^{\frac{5}{2}}(\sigma((1+\sigma)^2 + n^4) + (2+\sigma)n^4)}{(1+\sigma)^4 + 4(1+\sigma)^4 n^4} \\
&\geq \frac{2n^{\frac{5}{2}}n^4}{(1+\sigma)^4(1+4n^4)} = \frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}\frac{5n^4}{1+4n^4} \\
&\geq \frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}, \quad n \in \mathbb{N}.
\end{aligned}$$

By choosing an $c := \frac{5}{2}(\sigma^2+1)^{\frac{5}{8}}(1+\sigma)^4$ we obtain $|s_n|^{\frac{5}{4}} \leq c\frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}$ for every $n \geq 1$, and thus

$$\left|C_o(s_nE_o - A_o)^{-1}B_o\right| \geq \mathrm{Re}\left(C_o(s_nE_o - A_o)^{-1}B_o\right) \geq \frac{1}{c}|s_n|^{\frac{5}{4}}, \quad n \geq 1.$$

Hence, the growth of $C_o(s_nE_o - A_o)^{-1}B_o$ (and thus also that of the resolvent $(sE - A)^{-1}$) is more than quadratic along lines $\mathrm{Re}\, s = \sigma$ parallel to the imaginary axis for every $\sigma > 0$. Therefore, using Theorem 3.3 it follows that $p^{(E,A)}_{\mathrm{c,res}} = 3$.

Example 3.5. *(A class of well-posed systems with $p^{(E,A)}_{\mathrm{res}} \leq 2$) Let A_o with $\mathrm{dom}(A_o) \subset W$ generate a*

[1] This is a function that George Weiss (Tel Aviv) scribbled on paper for one of the authors. Thanks for that!

C_0-semigroup on W, where W is a Hilbert space. For $b \in W$ and $c \in \operatorname{dom}(A_o^*)$ with $\langle b,c\rangle \neq 0$ define the operators $Bu = bu$ where $u \in \mathbb{C}$ and $Cz = \langle z,c\rangle$ for any $z \in W$. We define the DAE on $Z = W \times \mathbb{C}$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{=:E} x(t) = \underbrace{\begin{bmatrix} A_o & B \\ C & 0 \end{bmatrix}}_{=:A} x(t), \quad t \geq 0. \tag{9}$$

Defining for $s \in \rho(A_o)$,

$$G(s) = \langle (sI - A_o)^{-1} b, c \rangle$$

we obtain

$$(sE-A)^{-1} = \begin{bmatrix} (sI-A_o)^{-1} - (sI-A_o)^{-1}BG(s)^{-1}C(sI-A_o)^{-1} & (sI-A_o)^{-1}BG(s)^{-1} \\ G(s)^{-1}C(sI-A_o)^{-1} & -G(s)^{-1} \end{bmatrix}.$$

Note that the condition $\langle b,c\rangle \neq 0$ implies $G(s) \neq 0$ for $s \in \rho(A_o)$. Thus, $\rho(E,A)$ is non-empty and the system is regular. Since

$$\lim_{s\to\infty} sG(s) = \langle c,b\rangle,$$

for large s, $G(s)^{-1} \leq Ms$ and so the resolvent index is at most 2.

4 Chain index

The concept of Wong sequences [34] is needed to define the chain index. For a set $V \subset X$ and operator $A : \operatorname{dom}(A) \subset X \to X$ let $A^{-1}(V)$ indicate the pre-image of V; that is, the set of all elements of $\operatorname{dom}(A)$ that map to V. For $X = Z = \mathbb{C}^n$ ($n \in \mathbb{N}$) we define

$$\begin{aligned} V_0 &:= \mathbb{C}^n, & V_{k+1} &:= A^{-1}(E(V_k)), & k &\in \mathbb{N}, \\ W_0 &:= \{0\}, & W_{k+1} &:= E^{-1}(A(W_k)), & k &\in \mathbb{N}. \end{aligned}$$

Thus, there exist $p_1, p_2 \in \mathbb{N}$ such that $V_{k+1} \subsetneq V_k$, $0 \leq k < p_1$, $W_k \subsetneq W_{k+1}$, $0 \leq k < p_2$ and $V_{p_1} = V_{p_1+k}$, $W_{p_2} = W_{p_2+k}$ for all $k \in \mathbb{N}$ [2, p. 3]. Let $x_{p_2} \in W_{p_2}$. Then, there exist a $x_{p_2-1} \in W_{p_2-1}$ with $Ex_{p_2} = Ax_{p_2-1}$. Inductively, one obtains

$$\begin{aligned} Ex_1 &= 0, \\ Ex_2 &= Ax_1, \\ &\vdots \\ Ex_{p_2} &= Ax_{p_2-1}, \end{aligned} \tag{10}$$

$x_k \in W_k$, $0 \leq k \leq p_2$, which is called a *chain of length p_2*. The concept of such a chain is not new and can be found in [2, p. 15] for the finite dimensional case or [31, p. 18] for arbitrary Banach spaces. It should be mentioned that the latter reference does not call (10) a chain.

Definition 4.1 (chain index).
We call $(x_1,\ldots,x_p) \in \operatorname{dom}(A)^{p-1} \times X$, $p \in \mathbb{N}$, a chain of (E,A) of length p, if $x_1 \in \ker E \setminus \{0\}$, $x_k \notin \ker A$, $k = 1,\ldots,p-1$, and

$$Ex_{k+1} = Ax_k, \quad k = 1,\ldots,p-1. \tag{11}$$

The chain index $p_{\text{chain}}^{(E,A)} \in \mathbb{N}_0$ of (E,A) is the supremum over all chain lengths of (E,A).

With this definition we exclude the case where the chain index can be infinite. Such an example is obtained when (E,A) has a form as in (2), whereby N is quasi-nilpotent, i.e. $\sigma(N) = \{0\}$. For such systems the chain index is not defined.

Proposition 4.2. *The chain index is, given that it exists, uniquely defined. To be more specific, let* $(E,A) \sim (\tilde{E},\tilde{A})$. *Then* $p_{\text{chain}}^{(E,A)} = p_{\text{chain}}^{(\tilde{E},\tilde{A})}$.

Proof. Since $(E,A) \sim (\tilde{E},\tilde{A})$ there exist two isomorphisms $P\colon X \to \tilde{X}$, $Q\colon Z \to \tilde{Z}$, such that

$$E = Q^{-1}\tilde{E}P \quad \text{and} \quad A = Q^{-1}\tilde{A}P.$$

Let $p_{\text{chain}}^{(E,A)} \in \mathbb{N}$. Thus, there exist a chain $(x_1,\ldots,x_{p_{\text{chain}}^{(E,A)}})$, such that

$$Ex_1 = 0, \quad Ex_2 = Ax_1, \quad \ldots, \quad Ex_{p_{\text{chain}}^{(E,A)}} = Ax_{p_{\text{chain}}^{(E,A)}-1}.$$

Since $E = Q^{-1}\tilde{E}P$, $A = Q^{-1}\tilde{A}P$ and Q is an isomorphism,

$$\tilde{E}Px_1 = 0, \quad \tilde{E}Px_2 = \tilde{A}Px_1, \quad \ldots, \quad \tilde{E}Px_{p_{\text{chain}}^{(E,A)}} = \tilde{A}Px_{p_{\text{chain}}^{(E,A)}-1}.$$

Because P is an isomorphism, $Px_1 \neq 0$ and thus $Px_1 \in \ker \tilde{E} \setminus \{0\}$. If we assume that $Px_k \in \ker \tilde{A}$ for any $1 \leq k \leq p_{\text{chain}}^{(E,A)} - 1$, then $0 = Q^{-1}\tilde{A}Px_k = Ax_k$, which would be a contradiction to $x_k \in X \setminus \ker A$. Therefore, $Px_k \notin \ker \tilde{A}$ for all $1 \leq k \leq p_{\text{chain}}^{(E,A)} - 1$. Hence, $(Px_1,\ldots,Px_{p_{\text{chain}}^{(E,A)}})$ denotes a chain of $(\tilde{E},\tilde{A})$ of length $p_{\text{chain}}^{(E,A)}$. Therefore, the chain index of $(\tilde{E},\tilde{A})$ is bounded from below by $p_{\text{chain}}^{(E,A)}$ and, equivalently, the chain index of (E,A) is bounded from below by $p_{\text{chain}}^{(\tilde{E},\tilde{A})}$. □

5 Radiality index

In this section we introduce a less well-known index, namely, the *radiality index*, and show that this index does not always exist. This index was originally known as *weak* (E,p)*-radiality* [31, p. 21]. Although not as widely known as some of the other indices, it is useful for infinite dimensions. For instance, with a few additional assumptions, it provides a means to decompose the DAE into a Weierstraß form; see [31, 15].

Definition 5.1 (radiality index).
The radiality index of (E,A) *is the smallest number* $p_{\text{rad}}^{(E,A)} \in \mathbb{N}_0$*, such that there exist a* $\omega \in \mathbb{R}$, $C > 0$ *with* $(\omega,\infty) \subseteq \rho(E,A)$ *and*

$$\begin{aligned} \left\| (\lambda_0 E - A)^{-1}E \cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}}E - A)^{-1}E \right\| &\leq C \prod_{k=0}^{p_{\text{rad}}^{(E,A)}} \frac{1}{|\lambda_k - \omega|}, \\ \left\| E(\lambda_0 E - A)^{-1} \cdot \ldots \cdot E(\lambda_{p_{\text{rad}}^{(E,A)}}E - A)^{-1} \right\| &\leq C \prod_{k=0}^{p_{\text{rad}}^{(E,A)}} \frac{1}{|\lambda_k - \omega|}, \end{aligned} \tag{12}$$

for all $\lambda_0,\ldots,\lambda_{p_{\text{rad}}^{(E,A)}} > \omega$. *The radiality index is called a complex radiality index, denoted by* $p_{\text{c,rad}}^{(E,A)}$*, if* $\mathbb{C}_{\text{Re}>\omega} \subseteq \rho(E,A)$ *and* (12) *holds for* $p_{\text{rad}}^{(E,A)}$.

Clearly, $p_{\text{rad}}^{(E,A)} \leq p_{\text{c,rad}}^{(E,A)}$. To get a better understanding for this definition one can start by looking at the radiality index with $p_{\text{rad}}^{(E,A)} = 0$, $X = Z$ and $E = I_X$. Then, (12) translates into the well known Hille-Yosida type estimate. This suggests that the radiality index implies further well-posedness results. In fact, as soon as slightly stronger assumptions are made, namely strong (E,p)-radiality [31, Sec. 2.5], one obtains the well-posedness of the system [31, Sec. 2.5 & 2.6].

Proposition 5.2. *The radiality index is unique. That is, if* $(E,A) \sim (\tilde{E},\tilde{A})$ *then* $p_{\text{rad}}^{(E,A)} = p_{\text{rad}}^{(\tilde{E},\tilde{A})}$ *and* $p_{\text{c,rad}}^{(E,A)} = p_{\text{c,rad}}^{(\tilde{E},\tilde{A})}$.

Proof. Since $(E,A) \sim (\tilde{E},\tilde{A})$ there exist two isomorphisms $P\colon X \to \tilde{X}$, $Q\colon Z \to \tilde{Z}$, such that

$$E = Q^{-1}\tilde{E}P \quad \text{and} \quad A = Q^{-1}\tilde{A}P.$$

Assume, that (E,A) has complex radiality index $p_{\text{c,rad}}^{(E,A)}$. Thus, there exist positive constants $C > 0$, $\omega > 0$, such that $\mathbb{C}_{\text{Re}>\omega} \subseteq \rho(E,A)$ and

$$\left\|(\lambda_0 E - A)^{-1}E \cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} E - A)^{-1}E\right\| \le \frac{C}{|(\lambda_0 - \omega)\cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} - \omega)|},$$
$$\left\|E(\lambda_0 E - A)^{-1} \cdot \ldots \cdot E(\lambda_{p_{\text{rad}}^{(E,A)}} E - A)^{-1}\right\| \le \frac{C}{|(\lambda_0 - \omega)\cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} - \omega)|},$$

for all $\lambda_0, \ldots, \lambda_{p_{\text{rad}}^{(E,A)}} \in \rho(E,A)$. Since

$$P(\lambda E - A)^{-1}Q^{-1} = (\lambda\tilde{E} - \tilde{A})^{-1}$$

for all $\lambda \in \rho(E,A)$ we derive $\mathbb{C}_{\text{Re}>\omega} \subseteq \rho(E,A) \subseteq \rho(\tilde{E},\tilde{A})$ and therefore

$$\begin{aligned}
&\left\|(\lambda_0 \tilde{E} - \tilde{A})^{-1}\tilde{E} \cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} \tilde{E} - \tilde{A})^{-1}\tilde{E}\right\| \\
&\qquad \le \|P\| \cdot \left\|(\lambda_0 E - A)^{-1}E \cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} E - A)^{-1}E\right\| \cdot \left\|P^{-1}\right\| \\
&\qquad \le \frac{\tilde{C}}{|(\lambda_0 - \omega)\cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} - \omega)|}, \\
&\left\|\tilde{E}(\lambda_0 \tilde{E} - \tilde{A})^{-1} \cdot \ldots \cdot \tilde{E}(\lambda_{p_{\text{rad}}^{(E,A)}} \tilde{E} - \tilde{A})^{-1}\right\| \\
&\qquad \le \|Q\| \cdot \left\|E(\lambda_0 E - A)^{-1} \cdot \ldots \cdot E(\lambda_{p_{\text{rad}}^{(E,A)}} E - A)^{-1}\right\| \cdot \left\|Q^{-1}\right\| \\
&\qquad \le \frac{\tilde{C}}{|(\lambda_0 - \omega)\cdot \ldots \cdot (\lambda_{p_{\text{rad}}^{(E,A)}} - \omega)|}
\end{aligned}$$

for all $\lambda_0, \ldots, \lambda_{p_{\text{rad}}^{(E,A)}} \in \mathbb{C}_{\text{Re}>\omega}$, for $\tilde{C} := 2C\max\{\|P\| \cdot \left\|P^{-1}\right\|, \|Q\| \cdot \left\|Q^{-1}\right\|\}$. Consequently, $(\tilde{E},\tilde{A})$ has at most radiality index $p_{\text{c,rad}}^{(E,A)}$. By an identical argument, (E,A) has at most radiality index $p_{\text{c,rad}}^{(\tilde{E},\tilde{A})}$. The equality $p_{\text{rad}}^{(E,A)} = p_{\text{rad}}^{(\tilde{E},\tilde{A})}$ follows analogously by restricting elements of the resolvent to (ω,∞). □

Example 5.3. *(A system with radiality index 0.) We consider the dynamics of a piezoelectric beam on an interval* $[a,b]$ *fixed at one end and free at the other end, modelled by the partial differential equations*

$$\rho \frac{\partial^2 v(\xi,t)}{\partial t^2} = \alpha \frac{\partial}{\partial \xi}\frac{\partial v(\xi,t)}{\partial \xi} - \gamma\beta \frac{\partial}{\partial \xi}\frac{\partial p(\xi,t)}{\partial \xi}, \tag{13a}$$
$$\mu \frac{\partial^2 p(\xi,t)}{\partial t^2} = \beta \frac{\partial}{\partial \xi}\frac{\partial p(\xi,t)}{\partial \xi} - \gamma\beta \frac{\partial}{\partial \xi}\frac{\partial v(\xi,t)}{\partial \xi}, \tag{13b}$$

where $v(\xi,t)$ *is the longitudinal displacement and* $p(\xi,t)$ *the electric charge for* $\xi \in [a,b[$ *and* $t > 0$. *In this context* $\rho > 0$ *stands for the material density,* $\mu \ge 0$ *the magnetic permeability,* $\alpha_1 > 0$ *the elastic stiffness,* $\beta > 0$ *the impermittivity and* $\gamma > 0$ *the piezoelectric coefficients. Defining*

$$\alpha := \alpha_1 + \gamma^2\beta$$

the boundary conditions are

$$v(a,t) = 0, \tag{14a}$$

$$p(a,t) = 0, \tag{14b}$$

$$\beta \frac{\partial p(b,t)}{\partial \xi} - \gamma\beta \frac{\partial v(b,t)}{\partial \xi} = 0, \tag{14c}$$

$$\alpha \frac{\partial v(b,t)}{\partial \xi} - \gamma\beta \frac{\partial p(b,t)}{\partial \xi} = 0. \tag{14d}$$

This model was shown to be a well-posed port-Hamiltonian system associated with a contraction semi-group [24]. However, there are quite a few choices for the state variables. We use here a different choice of state variable[2]

$$z(\xi,t) := \begin{bmatrix} z_1(\xi,t) \\ z_2(\xi,t) \\ z_3(\xi,t) \\ z_4(\xi,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ \sqrt{\mu}\frac{\partial p(\xi,t)}{\partial t} \end{bmatrix}.$$

Defining

$$Q := \begin{bmatrix} \alpha & 0 & -\gamma\beta & 0 \\ 0 & I & 0 & 0 \\ -\gamma\beta & 0 & \beta & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad E := \begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & 0 & 0 & \sqrt{\mu} \end{bmatrix}, \quad P_1 := \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix},$$

the PDAE can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} E z(\xi,t) = A z(\xi,t),$$

where $A = P_1 \frac{\partial}{\partial \xi} Q$. *This choice of state variables yields equations in the port-Hamiltonian formulation used in [16].*

Generally, μ *is very small and it is often taken to be zero, which yields the* quasi-static *piezo-electric beam. If* $\mu = 0$ *the operator* E *becomes singular and the fourth state variable becomes identically zero. The PDAE becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix} = P_1 \frac{\partial}{\partial \xi} Q \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix}.$$

Removing the fourth column of E *and* Q*, since the* 4th *variable is zero, we obtain*

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \sqrt{\rho} & 0 & 0 \\ 0 & \sqrt{\rho} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \end{pmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \frac{\partial}{\partial \xi} \begin{bmatrix} \alpha & 0 & -\gamma\beta \\ 0 & I & 0 \\ -\gamma\beta & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \end{pmatrix}.$$

This PDAE fits into the class of systems described in [15, Sec. IV]. Consequently, it has radiality index 0.

Next, we compare the radiality index with the resolvent index.

[2] This was suggested by Hans Zwart.

Proposition 5.4. *If the (complex) radiality index $p_{\text{rad}}^{(E,A)}$ exists, then the (complex) resolvent index also exists. Furthermore, $p_{\text{rad}}^{(E,A)}+1 \geq p_{\text{res}}^{(E,A)}$ ($p_{\text{c,rad}}^{(E,A)}+1 \geq p_{\text{c,res}}^{(E,A)}$).*

Proof. We have to show that there exist a $C>0$, $\omega>0$, such that $(\omega,\infty) \subseteq \rho(E,A)$ ($\mathbb{C}_{\text{Re}>\omega} \subseteq \rho(E,A)$) and (3) holds for all $\lambda \in (\omega,\infty)$ ($\lambda \in \mathbb{C}_{\text{Re}>\omega}$) and a $p \leq p_{\text{rad}}+1$ ($p \leq p_{\text{c,rad}}+1$). The first part of this statement is implied by existence of the radiality index. Hence, we only have to prove (3). An operator A is (E,p)-sectorial [31, sec. 3.1] if the inequalities (12) hold on a sector $\{\, \mu \in \mathbb{C} \mid |\arg(\mu-\omega)| < \theta, \mu \neq \omega \,\}$, for some $\omega \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2},\pi)$. Following he proof of [31, Lem. 3.1.1] for positive $\lambda > \omega$ (for $\lambda \in \mathbb{C}_{\text{Re}>\omega}$), completes the proof of this proposition.

□

The following example shows that in general existence of the resolvent index does not imply the existence of the radiality index.

Example 5.5. *(A system where the resolvent index exists, but the radiality index does not.) Let $X = L^2(0,\infty)\times\mathbb{R}$ and define*

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \partial_\xi & 0 \\ \delta_0 & 1 \end{bmatrix}$$

with $\operatorname{dom}(A) := H^1(0,\infty)\times\mathbb{R}$*, where* $\delta_0\colon L^2(0,\infty)\to\mathbb{R}$, $x\mapsto x(0)$*. Let* $\lambda>0$ *and* $\begin{pmatrix} f \\ g\end{pmatrix} \in X$*. Then*

$$(\lambda E - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

if and only if

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_\cdot^\infty e^{\lambda(\cdot-s)} f(s)\,ds \\ -g - \int_0^\infty e^{-\lambda s} f(s)\,ds \end{pmatrix}.$$

Hence,

$$(\lambda E - A)^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \int_\cdot^\infty e^{\lambda(\cdot-s)} f(s)\,ds \\ -g - \int_0^\infty e^{\lambda(0-s)} f(s)\,ds \end{pmatrix}$$

and $\left\|(\lambda E-A)^{-1}\right\| \leq M = M\lambda^0$ *for a* $M>0$ *for all* $\lambda \geq 0$*. Thus,* (E,A) *has resolvent index* $p_{\text{res}}^{(E,A)} = 1$*. Furthermore, we have*

$$\left((\lambda E-A)^{-1}E\right)^{p+1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \int_\cdot^\infty \left(\int_{s_{p+1}}^\infty \left(\ldots\left(\int_{s_2}^\infty e^{\lambda(\cdot-s_1)} f(s_1)\,ds_1\right)\ldots\right) ds_p\right) ds_{p+1} \\ -\int_0^\infty \left(\int_{s_{p+1}}^\infty \left(\ldots\left(\int_{s_2}^\infty e^{\lambda(0-s_1)} f(s_1)\,ds_1\right)\ldots\right) ds_p\right) ds_{p+1} \end{pmatrix}.$$

Let $f(\tau) = \tau e^{-\lambda\tau}$*. Then,* $\|f\|_{L^2} = \frac{1}{\lambda^{\frac{3}{2}}}$ *and*

$$\begin{aligned} \left\|\left((\lambda E-A)^{-1}E\right)^{p+1}\right\| &\geq \left\|\left((\lambda E-A)^{-1}E\right)^{p+1}\begin{pmatrix} \lambda^{\frac{3}{2}} f \\ 0 \end{pmatrix}\right\| \\ &\geq \left|\int_0^\infty \left(\int_{s_{p+1}}^\infty \left(\cdots\left(\int_{s_2}^\infty e^{-2\lambda s_1} s_1\,ds_1\right)\right) ds_p\right) ds_{p+1}\right| \\ &= \frac{p+1}{2^{p+1}} \frac{1}{\lambda^{p+1-\frac{3}{2}}} \end{aligned} \tag{15}$$

for every $p \in \mathbb{N}$. Consequently, the radiality index does not exist. Because if it exists, then there would exist $C > 0$, $\omega > 0$, such that

$$\left\| \left((\lambda E - A)^{-1} E \right)^{p+1} \right\| \leq C \frac{1}{(\lambda - \omega)^{p+1}}$$

for all $\lambda > \omega$, which would contradict (15).

6 Nilpotency index

Next, we are going to look at what is probably the best known index, the *nilpotency index* (also known as the *Weierstraß-index*). As the alternative term suggests, the most important part of the definition is the existence of the Weierstraß form. For the sake of simplicity we exclude the case that X and Z are 0-dimensional.

Definition 6.1 (nilpotency index).
Assume, that the DAE (1) has a Weierstraß form given by (2). Then, the nilpotency index of (E,A), denoted by $p_{\text{nilp}}^{(E,A)} \in \mathbb{N}_0$, is the nilpotency degree of N if it is present and 0 if N is absent. In the latter case one has $(E,A) \sim (I_{Y^1}, A_1)$ in (2).

Here, one should keep in mind that we are only dealing with regular (E,A) (as mentioned in the introduction) and stick to the Definition of a Weierstraß form as seen in Section 2. It is possible to extend the definition to non-regular (E,A) using the Kronecker form [17, Thm. 2.3]. Further studies about the existence of a Weierstraß form or the regularity of (E,A) can be found [9, 11, 15, 20, 23, 31].

Proposition 6.2. *Assume that the DAE (1) has a Weierstraß form. Let $\lambda \in \mathbb{C}$ such that $\lambda E - A$ is bijective (which exists due to our regularity assumption on (1)). Then the nilpotency index of (E,A) is is the smallest number $k \in \mathbb{N}$, such that*

$$\ker \left((\lambda E - A)^{-1} E \right)^k = \ker \left((\lambda E - A)^{-1} E \right)^{k+1}.$$

In particular, the nilpotency index is well-defined.

Proof. Assume that $P\colon X \to Y^1 \times Y^2$, $Q\colon Z \to Y^1 \times Y^2$, such that

$$E = Q^{-1} \begin{bmatrix} I_{Y^1} & 0 \\ 0 & N \end{bmatrix} P, \quad A = Q^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & I_{Y^2} \end{bmatrix} P. \tag{16}$$

Then the result follows, since for all $k \in \mathbb{N}$,

$$\left((\lambda E - A)^{-1} E \right)^k = P^{-1} \begin{bmatrix} (\lambda I_{Y^1} - A_1)^{-k} & 0 \\ 0 & (\lambda N - I_{Y^2})^{-k} N^k \end{bmatrix} P.$$ □

According to this definition, the nilpotency index is always a natural number (including 0). This means that the nilpotency index can never be ∞, since we require a nilpotent operator N in (2). To handle this more general situation, one would have to replace the nilpotency of N with *quasi-nilpotency*, namely with $\sigma(N) = \{0\}$. For example, $N\colon \ell^2 \to \ell^2$, $(x_1, x_2, , x_3, \ldots) \mapsto (0, \frac{x_1}{2^1}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \ldots)$.

A disadvantage of the nilpotency index is that it requires the Weierstraß form. In finite dimensions one only needs the system to be regular to obtain such a form, see, for instance, [17, Def. 2.9] for a constructive procedure. There is no standard procedure to obtain the Weierstraß form for infinite dimensional systems, and, in fact, it has not been proven that such a form always exists.

The following Proposition compares the nilpotency index with the index terms introduced so far. First comparisions of the resolvent index and the nilpotency index, can be found in [4, Chapter 7].

Proposition 6.3. *(i) If the nilpotency index exists, then the chain index also exists with* $p_{\text{nilp}}^{(E,A)} = p_{\text{chain}}^{(E,A)}$.

(ii) If the nilpotency index and resolvent index exist, then $p_{\text{res}}^{(E,A)} \geq p_{\text{nilp}}^{(E,A)}$.

(iii) If the nilpotency index and radiality index exist, then $p_{\text{rad}}^{(E,A)} + 1 \geq p_{\text{nilp}}^{(E,A)}$.

Proof. We start with the proof of Part (i). Since $(E,A) \sim (\tilde{E},\tilde{A}) := \left(\begin{bmatrix} I_{Y^1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{Y^2} \end{bmatrix} \right)$ for a Hilbert space $Y = Y^1 \times Y^2$, where N has nilpotency degree $p_{\text{nilp}}^{(E,A)}$, there exist two isomorphisms $P\colon X \to Y^1 \times Y^2$, $x \mapsto (P_1 x, P_2 x)$, $Q\colon Z \to Y^1 \times Y^2$, $z \mapsto (Q_1 z, Q_2 z)$, such that (16) holds. For simplicity we will show, that $p_{\text{nilp}}^{(\tilde{E},\tilde{A})} = p_{\text{chain}}^{(\tilde{E},\tilde{A})}$ (because then the rest follows from Proposition 4.2 and 6.2). Let $p := p_{\text{nilp}}^{(\tilde{E},\tilde{A})}$ and let $x_k = \binom{x_{k,1}}{x_{k,2}}$, $k = 1,\ldots,q$ denote an arbitrary chain of length $q-1$. Then $x_{k,1} = 0$ for all $k = 1,\ldots,q$ and

$$\tilde{E}x_1 = 0, \quad \tilde{E}x_2 = \tilde{A}x_1, \quad \ldots, \quad \tilde{E}x_q = \tilde{A}x_{q-1}.$$

Hence $Nx_{l,2} = x_{l-1,2}$ for all $2 \leq l$ and $x_{1,2} = N^1 x_{2,2} = \ldots = N^{q-1} x_{q,2}$. Since $x_{1,2} \neq 0$, $q > p$ is not possible. Thus, the existence of a Weierstraß form implies that the chain index also exists and that it is at most $p = p_{\text{nilp}}^{(E,A)}$. In order to prove that these two index-terms are equal, we need to show that there does exist a chain of length p. This follows directly by choosing $x_k := \binom{x_{k,1}}{x_{k,2}} = \binom{0}{N^{p-k}z}$, $k = 1,\ldots,p$, for a $z \in Y^2$ such that $N^p z = 0$ and $N^{p-1} z \neq 0$.

We now prove Part (ii). From the definition of the resolvent index, there exist $C > 0, \omega > 0$, such that $(\omega,\infty) \subseteq \rho(E,A)$ and

$$\left\|(\lambda E - A)^{-1}\right\| \leq C\,|\lambda|^{p_{\text{res}}^{(E,A)}-1}.$$

By Proposition 3.2, for some $\tilde{C} > 0$

$$\left\| \begin{bmatrix} (\lambda I_{Y^1} - A_1)^{-1} & 0 \\ 0 & (\lambda N - I_{Y^2})^{-1} \end{bmatrix} \right\| \leq \tilde{C}\,|\lambda|^{p_{\text{res}}^{(E,A)}-1} \tag{17}$$

for all $\lambda > \omega$. Since N is nilpotent with nilpotency degree p we have $(\lambda N - I_{Y^2})^{-1} = -\sum_{i=0}^{p-1} (\lambda N)^i$ and, together with (17), we derive

$$\begin{aligned} \left\|(\lambda N - I_{Y^2})^{-1}\right\| &= \lambda^{p-1} \left\| I_{X^2} \frac{1}{\lambda^{p-1}} + N \frac{1}{\lambda^{p-2}} + \ldots + N^{p-2} \frac{1}{\lambda} + N^{p-1} \right\| \\ &\leq \tilde{C} \lambda^{p_{\text{res}}^{(E,A)}-1} \end{aligned}$$

for all $\lambda > \omega$ and therefore $p_{\text{nilp}}^{(E,A)} = p \leq p_{\text{res}}^{(E,A)}$.

Finally, in order to prove Part (iii) we assume that the radiality index $p_{\text{rad}}^{(E,A)}$ exists. That is, there are $C > 0$ and $\omega > 0$ such that

$$\begin{aligned} \left\| ((\lambda N - I_{Y^2})^{-1} N)^{p_{\text{rad}}^{(E,A)}+1} \right\| &\leq \left\| ((\lambda \tilde{E} - \tilde{A})^{-1} \tilde{E})^{p_{\text{rad}}^{(E,A)}+1} \right\| \\ &\leq \left\|P^{-1}\right\| \left\| ((\lambda E - A)^{-1} E)^{p_{\text{rad}}^{(E,A)}+1} \right\| \|P\| \\ &\leq \frac{C}{(\lambda - \omega)^{p_{\text{rad}}^{(E,A)}+1}} \end{aligned}$$

for all $\lambda > \omega$. Now, $((\lambda N - I_{Y^2})^{-1} N)^{p_{\text{rad}}^{(E,A)}+1} = \frac{1}{\lambda^{p_{\text{rad}}^{(E,A)}+1}} \left(-\sum_{k=1}^{p-1} (\lambda N)^k \right)^{p_{\text{rad}}^{(E,A)}+1}$ is a polynomial of degree

$1 \le \deg \le (p-1)$. Thus,

$$\left\| \left(-\sum_{k=1}^{p-1} (\lambda N)^k \right)^{p_{\mathrm{rad}}^{(E,A)}+1} \right\| \le C \frac{\lambda^{p_{\mathrm{rad}}^{(E,A)}+1}}{(\lambda-\omega)^{p_{\mathrm{rad}}^{(E,A)}+1}}. \tag{18}$$

The left-hand-side contains terms of the form $(\lambda N)^n$ where $n \ge p_{\mathrm{rad}}^{(E,A)} + 1$. Since it is bounded by the right-hand-side of (18) which is bounded as $\lambda \to \infty$, $p_{\mathrm{nilp}}^{(E,A)} = p \le p_{\mathrm{rad}}^{(E,A)} + 1$, as was to be proven. □

Remark 6.4. *For non-negative and self-adjoint $E \in \mathscr{L}(X)$ and dissipative $A\colon \operatorname{dom}(A) \subseteq X \to X$ with $(\omega,\infty) \subseteq \rho(E,A)$ (for a $\omega > 0$) the nilpotency index and the chain index is at most 2. This follows directly from Theorem 3.3 and Proposition 6.3.*

Example 6.5. *(A system with $p_{\mathrm{rad}}^{(E,A)} = 1$ and $p_{\mathrm{nilp}}^{(E,A)} = 2$) Let us recall Example 3.5. Defining*

$$\begin{aligned} R_{1,s} &= (sI-A_o)^{-1}(I - BG(s)^{-1}C(sI-A_o)^{-1}), \\ R_{2,s} &= G(s)^{-1}C(sI-A_o)^{-1}, \\ L_{2,s} &= (sI-A_o)^{-1}BG(s)^{-1} \end{aligned}$$

one has

$$\begin{aligned} (sE-A)^{-1}E &= \begin{bmatrix} R_{1,s} & 0 \\ R_{2,s} & 0 \end{bmatrix}, \\ E(sE-A)^{-1} &= \begin{bmatrix} R_{1,s} & L_{2,s} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

For the rest of this example we will focus on a particular choice of A_o, b and c: Let $W = L^2(0,1)$,

$$A_o z = z'', \quad \operatorname{dom}(A_o) = \left\{ \, w \in H^2(0,1) \;\middle|\; z'(0) = z'(1) = 0 \, \right\}, \quad b = c = \mathbb{1}, \tag{19}$$

where $\mathbb{1}$ is the constant 1-function. With these definitions of A_o, B and C,

$$\begin{aligned} (sI-A_o)^{-1}Bu &= \frac{1}{s}\mathbb{1}u, \\ C(sI-A_o)^{-1}z &= \frac{1}{s}\langle z, \mathbb{1}\rangle, \\ G(s) = C(sI-A_o)^{-1}B &= \frac{1}{s} \end{aligned}$$

and so $R_{1,s}z = (sI-A_o)^{-1}z - \mathbb{1}\langle z,1\rangle\frac{1}{s}$, $R_{2,s}z = \langle z,\mathbb{1}\rangle$, $L_{2,s}u = \mathbb{1}u$. Since $R_{2,s}$ is independent of s, the radiality degree must be larger than 0.

Define the projection onto $W_1 := \ker C \subset W$,

$$Q_c z := z - \frac{\langle z,c\rangle}{\langle c,c\rangle} c.$$

Then, with $W_2 := \operatorname{span} c$, Q_c splits W into $W_1 \oplus W_2$. It is easy to see that $R_{1,s}\alpha\mathbb{1} = 0$ for $\alpha\mathbb{1} \in \operatorname{span} c$ and

$R_{1,s}z = (sI - A_o)^{-1}z \in \ker C$ *for* $z \in \ker C$. *Thus, for* $w = \alpha z \mathbb{1} \in W_1 \oplus W_2$

$$(sE-A)^{-1}E(\mu E-A)^{-1}E\begin{pmatrix} z+\alpha\mathbb{1} \\ u\end{pmatrix} = \begin{bmatrix} R_{1,s}R_{1,\mu} & 0 \\ R_{2,s}R_{1,\mu} & 0\end{bmatrix}\begin{pmatrix} z+\alpha\mathbb{1} \\ u\end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu}z \\ 0\end{pmatrix},$$
$$E(sE-A)^{-1}E(\mu E-A)^{-1}\begin{pmatrix} z+\alpha\mathbb{1} \\ u\end{pmatrix} = \begin{bmatrix} R_{1,s}R_{1,\mu} & R_{2,s}L_{2,\mu} \\ 0 & 0\end{bmatrix}\begin{pmatrix} z+\alpha\mathbb{1} \\ u\end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu}z \\ 0\end{pmatrix}.$$

Since A_o *is the generator of a contraction semigroup,*

$$\|R_{1,s}R_{1,\mu}\| \leq \frac{1}{s\mu}, \quad s,\mu > 0.$$

Hence, the radiality index of (E,A) *is* 1.

Furthermore, it is possible to rewrite (E,A) *into a Weierstraß form with* $p_{\text{nilp}}^{(E,A)} = 2$*. For more concise notation, define* $\tilde{C}u := \frac{1}{\langle c,c\rangle}cu$*. We can define the isomorphisms* $U : W \times \mathbb{C} \to W_1 \times W_2 \times \mathbb{C}$ *and* $V : W_1 \times W_2 \times \mathbb{C} \to W \times \mathbb{C}$

$$U := \begin{bmatrix} Q_c & 0 \\ 0 & \tilde{C} \\ C & 0\end{bmatrix}, \qquad V := \begin{bmatrix} I & I & 0 \\ 0 & 0 & I\end{bmatrix}$$

which have inverses

$$U^{-1} := \begin{bmatrix} I & A_o & B \\ 0 & C & 0\end{bmatrix}, \qquad V^{-1} := \begin{bmatrix} Q_c & 0 \\ I-Q_c & 0 \\ 0 & I\end{bmatrix}.$$

These mappings will be applied to (9) *to obtain a splitting of the system into equations on* $W_1 \times W_2 \times \mathbb{C}$*. Noting that*

- *if* $z \in W_1$, $A_o z \in W_1$,
- $\tilde{C}C = I - Q_c$, $= 0$ *on* $\ker C$, I *on* $\operatorname{span} c$,
- $CB = 1$,

the isomorphisms U *and* V *lead to*

$$\tilde{E} := U\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}V = \begin{bmatrix} Q_c & Q_c & 0 \\ 0 & 0 & 0 \\ \tilde{C} & \tilde{C} & 0\end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{C} & 0\end{bmatrix},$$
$$\tilde{A} := U\begin{bmatrix} A_o & B \\ C & 0\end{bmatrix}V = \begin{bmatrix} Q_cA_o - Q_cA_o\tilde{C}C & Q_cA_o - Q_cA_o\tilde{C}C & 0 \\ \tilde{C}C & \tilde{C}C & 0 \\ \tilde{C}(A_o) & \tilde{C}A_o & 1\end{bmatrix} = \begin{bmatrix} A_o & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1\end{bmatrix}.$$

Define

$$N = \begin{bmatrix} 0 & 0 \\ \tilde{C} & 0\end{bmatrix}, \qquad \tilde{A}_o z = z''$$

with domain $\operatorname{dom}(A_o) \cap \ker C$*, which is dense in* $\ker C$ *[25]. The operator* A_o *will generate a* C_0*-semigroup on* $W_1 = \ker C$*. Via the isomorphisms* U *and* V *the system* (9) *is equivalent to the Weierstraß form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{W_1} & 0 \\ 0 & N\end{bmatrix}\begin{pmatrix} z_1(t) \\ z_2(t)\end{pmatrix} = \begin{bmatrix} \tilde{A}_o & 0 \\ 0 & I_{W_2\times\mathbb{C}}\end{bmatrix}\begin{pmatrix} z_1(t) \\ z_2(t)\end{pmatrix}. \tag{20}$$

A simple calculation shows that $N^2 = 0$*, and thus, the nilpotency index of the system is* 2.

Finding a Weierstraß form for the system (9) is identical to the question of establishing the zero dynamics of a system with state-space realization (A_o, B, C). That is, finding the largest space W_1 on which for initial conditions in W_1, the dynamics remain in W_1. Clearly $W_1 \subseteq \ker C$. In this case, the largest space is $W_1 = \ker C$. A more general situation with $c \in \operatorname{dom}(A_o^*)$ and $b \neq c$ is described in [35]. The case where $c \notin \operatorname{dom}(A_o^*)$ is treated in [25]. A similar type of construction was shown to hold for a class of boundary control systems with collocated observation; that is, the unbounded generalization of $\langle b, c\rangle \neq 0$ in [28], and for diffusion problems on an interval in [3].

Example 6.6. *(A system where the nilpotency index exists, but resolvent index does not.) Let $A_0 = -\partial_x^2 + ix$ be the complex Airy operator on $X^1 = L^2(\mathbb{R})$ with*

$$\operatorname{dom}(A_0) = \left\{ u \in H^2(\mathbb{R}) \;\middle|\; xu \in L^2(\mathbb{R}) \right\}.$$

By [14, 1] we have $\sigma(A_0) = \emptyset$ and there exist a constant $C > 0$, such that

$$\left\|(\lambda I_{X^1} - A_0)^{-1}\right\| = C(\operatorname{Re}\lambda)^{-\frac{1}{4}}\, \mathrm{e}^{\frac{4}{3}(\operatorname{Re}\lambda)^{\frac{3}{2}}}$$

for all $\operatorname{Re}\lambda > 0$. Thus, $\left\|(\lambda I_{X^1} - A_0)^{-1}\right\| \to \infty$ for $\operatorname{Re}\lambda \to \infty$. Let $N\colon X^2 \to X^2$ be a nilpotent operator of degree $p \in \mathbb{N}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\begin{bmatrix} I_{X^1} & 0 \\ 0 & N \end{bmatrix}}_{=E} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{bmatrix} A_0 & 0 \\ 0 & I_{X^2} \end{bmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

defines a DAE with nilpotency index $p = p_{\text{nilp}}^{(E,A)}$, but since $\left\|(\lambda I_{X^1} - A_0)^{-1}\right\|$ grows exponentially the resolvent index does not exist.

Example 6.7. *(A system where the nilpotency index exists, but radiality index does not.) Example 6.6 shows that in general the existence of the nilpotency index does not imply the existence of the radiality index.*

Example 6.8. *(A system where the resolvent index exists, but nilpotency index does not.) Consider the system*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x_1(t,\xi) &= -\tfrac{\partial}{\partial\xi}x_1(t,\xi), && (21)\\ 0 &= -x_1(t,0), && (22)\\ 0 &= -x_1(t,1) + x_2(t), && (23)\\ \tfrac{\mathrm{d}}{\mathrm{d}t}x_2(t) &= x_3(t), && (24) \end{aligned}$$

which corresponds to a differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ with

$$\begin{aligned} &X = L^2(0,1) \times \mathbb{C}^2, \quad \operatorname{dom}(A) = H^1(0,1) \times \mathbb{C}^2,\\ &Z = L^2(0,1) \times \mathbb{C}^3, \end{aligned} \tag{25}$$

and

$$E = \begin{bmatrix} I_{L^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{\partial}{\partial\xi} & 0 & 0 \\ -\delta_0 & 0 & 0 \\ -\delta_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

where $\delta_\xi \in H^1(0,1)^*$ is the evaluation operator at $\xi \in [0,1]$. The resolvent fulfills, for all $x_1, x_2, x_2 \in \mathbb{C}$,

$x_2 \in L^2(0,1)$,

$$(\lambda E - A)^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-s} x_3 + \int_0^1 \mathrm{e}^{-s(1-\xi)} f(\xi)\,\mathrm{d}\xi \\ s\,\mathrm{e}^{-s} x_3 + s\int_0^1 \mathrm{e}^{-s(1-\xi)} f(\xi)\,\mathrm{d}\xi - x_2 \\ \mathrm{e}^{-s\cdot} x_3 + \int_0^{\cdot} \mathrm{e}^{-s(\cdot-\xi)} f(\xi)\,\mathrm{d}\xi \end{pmatrix},$$

which gives $p_{\mathrm{res}}^{(E,A)} = 2$. *It has however been shown in [30] that the system has no nilpotency index.*

As we have already seen through Example 5.5 and 6.6 the nilpotency index, the resolvent index and the radiality index do not have to coincide. This is different in finite dimensions as we will see next.

Proposition 6.9. *Let X and Z be finite dimensional. Then* $p_{\mathrm{nilp}}^{(E,A)} = p_{\mathrm{rad}}^{(E,A)} + 1 = p_{\mathrm{res}}^{(E,A)}$.

Proof. In [10, Lem. 2.1] it is shown that $p_{\mathrm{nilp}}^{(E,A)} = p_{\mathrm{res}}^{(E,A)}$. Thus it remains to show $p_{\mathrm{nilp}}^{(E,A)} = p_{\mathrm{rad}}^{(E,A)} + 1$.

By Proposition 6.3 (iii) we already have $p_{\mathrm{rad}}^{(E,A)} + 1 \geq p_{\mathrm{nilp}}^{(E,A)}$. Let $\left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$ be the Weierstraß form of (E,A). For $\lambda \in \rho(E,A)$ we have

$$R_\lambda := \left(\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)^{-1} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} (\lambda I - J)^{-1} & 0 \\ 0 & (\lambda N - I)^{-1} N \end{bmatrix} \tag{27}$$

and since N commutes with $(\lambda N - I)^{-1} = -\sum_{k=0}^{p_{\mathrm{nilp}}^{(E,A)}-1} (\lambda N)^k$ this is the same as

$$L_\lambda := \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)^{-1}.$$

Let $\lambda_0, \ldots, \lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} \in \rho(E,A)$. Using the nilpotency of N we get

$$(\lambda_0 N - I)^{-1} N \cdot \ldots \cdot (\lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} N - I)^{-1} N = (\lambda_0 N - I)^{-1} \cdot \ldots \cdot (\lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} N - I)^{-1} N^{p_{\mathrm{nilp}}^{(E,A)}} = 0. \tag{28}$$

Thus,

$$R_{\lambda_0} \cdot \ldots \cdot R_{\lambda_{p_{\mathrm{nilp}}^{(E,A)}-1}} = \begin{bmatrix} \prod_{k=0}^{p_{\mathrm{nilp}}^{(E,A)}-1} (\lambda_k I - J)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since J is in Jordan form there exist a $C > 0$, such that for all $\lambda_0, \ldots, \lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} > \omega$ we have

$$\left\| (\lambda_0 I - J)^{-1} \cdot \ldots \cdot (\lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} I - J)^{-1} \right\| \leq K \frac{1}{(\lambda_0 - \omega) \cdot \ldots \cdot (\lambda_{p_{\mathrm{nilp}}^{(E,A)}-1} - \omega)}. \tag{29}$$

Thus, (E,A) has at most radiality index $p_{\mathrm{nilp}}^{(E,A)} - 1$ and therefore $p_{\mathrm{nilp}}^{(E,A)} \geq p_{\mathrm{rad}}^{(E,A)} + 1$. □

7 Differentiation index

The differentiation index is based on taking formal derivatives up to a certain order which leads to an ordinary differential equation. For details, see, for instance, [18, Sec. 3.10] and [17, Sec. 3.3]. Here we present a generalization of this concept to abstract differential-algebraic equations of the form (1) with the restriction that only the case where $f \equiv 0$ is considered. Using the fact that the operators E and A do not depend on time, the k-th formal derivative of (1) with $f = 0$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} E \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t) = A \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t).$$

The collection of the first μ formal derivatives of (1) is

$$\underbrace{\begin{bmatrix} A & -E & & & \\ & A & -E & & \\ & & \ddots & \ddots & \\ & & & A & -E \end{bmatrix}}_{=:M_\mu} \begin{pmatrix} x(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} x(t) \\ \vdots \\ \frac{\mathrm{d}^\mu}{\mathrm{d}t^\mu} x(t) \\ \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}t^{\mu+1}} x(t) \end{pmatrix} = 0 \tag{30}$$

which will be referred to as a *derivative array of order* μ. The operator M_μ maps from $\mathrm{dom}(A)^{\mu+1} \times X$ to $Z^{\mu+1}$.

Definition 7.1 (Differentiation index).
Let X, Z be Banach spaces, and let a linear differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ be given, where $E \in \mathscr{L}(X,Z)$, $(A, \mathrm{dom}(A))$ is a closed and densely defined linear operator from X to Z. The DAE $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ has differentiation index $p_{\mathrm{diff}}^{(E,A)} \in \mathbb{N}_0$, if the following holds:

(i) There exist a Banach space $\tilde{X}$, such that $\tilde{X}$ is densely embedded into X, an operator $(S, \mathrm{dom}(S))$ which is the generator of a strongly continuous semigroup on $\tilde{X}$, and the operator $M_{p_{\mathrm{diff}}^{(E,A)}}$ as defined in (30) satisfies

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p_{\mathrm{diff}}+1} \end{pmatrix} \in \ker M_{p_{\mathrm{diff}}^{(E,A)}} \quad \Longrightarrow x_0 \in \mathrm{dom}(S) \text{ with } x_1 = Sx_0.$$

(ii) The number $p_{\mathrm{diff}}^{(E,A)}$ is minimal with the properties in (a).

We call

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Sx(t) \tag{31}$$

an abstract completion ODE of (1).

Before establishing a connection between the solutions of the abstract completion ODE and the DAE itself, we introduce a lemma that characterizes solutions. This result is a direct consequence of the integration by parts formula. The proof is therefore omitted.

Lemma 7.2. *Let X, Z be Banach spaces, and let a linear differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ be given, where $E \in \mathscr{L}(X,Z)$, $(A, \mathrm{dom}(A))$ is a closed and densely defined linear operator from X to Z.*

(i) If $x\colon [0,\infty) \to X$ solves $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$, then for all test functions $\varphi\colon [0,\infty) \to \mathbb{R}$ (i.e., those which are smooth and have compact support in $(0,\infty)$),

$$-\int_{\mathbb{R}_{\geq 0}} \frac{\mathrm{d}}{\mathrm{d}t} \varphi(t) Ex(t)\, \mathrm{d}t = \int_{\mathbb{R}_{\geq 0}} \varphi(t) Ax(t)\, \mathrm{d}t\,. \tag{32}$$

(ii) Conversely, if $x\colon [0,\infty) \to \mathbb{R}$ fulfills (32) for all test functions $\varphi\colon [0,\infty) \to \mathbb{R}$, and it is continuously differentiable with $x(t) \in \mathrm{dom}(A)$ for all $t \geq 0$, then it is a solution of $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$.

Theorem 7.3. *Let X,Z be Banach spaces, and let $E \in \mathscr{L}(X,Z)$, and $(A, \mathrm{dom}(A))$ be a closed and densely defined linear operator from X to Z. Assume that the linear differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ has differentiation index $p_{\mathrm{diff}}^{(E,A)}$. If x solves $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$, then it fulfills the abstract completion ODE (31).*

Proof. Assume that x solves $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$. Lemma 7.2 implies that for all test functions $\varphi\colon [0,\infty) \to \mathbb{R}$ one has (32). Now testing with $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\varphi(t)$, $k = 0,\dots,p_{\mathrm{diff}}^{(E,A)}+1$, we obtain that the vectors

$$x_k = (-1)^k \int_{\mathbb{R}_{\geq 0}} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\varphi(t)x(t)\,\mathrm{d}t\,, \quad k = 0,\dots,p_{\mathrm{diff}}^{(E,A)}+1,$$

satisfy

$$\begin{bmatrix} A & -E & & & \\ & A & -E & & \\ & & \ddots & \ddots & \\ & & & A & -E \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p_{\mathrm{diff}}^{(E,A)}} \\ x_{p_{\mathrm{diff}}^{(E,A)}+1} \end{pmatrix} = -0,$$

and the definition of the differentiation index yields

$$x_1 = Sx_0.$$

Thus, for all test functions $\varphi\colon [0,\infty) \to \mathbb{R}$

$$-\int_{\mathbb{R}_{\geq 0}} \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t)x(t)\,\mathrm{d}t = \int_{\mathbb{R}_{\geq 0}} \varphi(t)Sx(t)\,\mathrm{d}t\,.$$

Another application of Lemma 7.2 now shows that x is a solution of the abstract completion ODE of (1). □

Remark 7.4. *Let X,Z be Banach spaces, and let $E \in \mathscr{L}(X,Z)$, and $(A, \mathrm{dom}(A))$ be a closed and densely defined linear operator from X to Z. Assume that the linear differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ has differentiation index $p_{\mathrm{diff}}^{(E,A)}$, and let (31) be an abstract completion ODE. Let $x_0 \in X$, such that the differential-algebraic initial value problem $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$, $Ex_0 = Ex(0)$ has a solution x. Since the solution of the abstract completion boundary control system (31) with initial value $x(0) = x_0 \in X$ is unique, we can conclude that x is the unique solution of the differential-algebraic initial value problem $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$.*

Example 7.5. *(A system with $p_{\mathrm{diff}}^{(E,A)} = 2$) Once again, consider the system (21)–(24), which corresponds to a DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ with spaces (25) and operators as in (26). Before we determine the differentiation index, we analyse the solution behavior and we particularly determine the set of $x_0 = (x_{01}, x_{02}, x_{03})$, with $x_{01} \in L^2([0,1])$ and $x_{02}, x_{03} \in \mathbb{C}$, such that $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ has a solution with initial value $x(0) = x_0$. To this end we first note that the solution of the boundary value problem (21), (22) with $x_1(0) = x_{10}$ reads $x_1\colon [0,\infty) \to L^2(0,1)$ with*

$$(x_1(t))(\xi) = x_1(t,\xi) = \begin{cases} x_{10}(\xi - t): & \xi \geq t, \\ 0: & \xi < t. \end{cases}$$

By then substituting this into (22) and (23), we obtain that

$$x_2(t) = \begin{cases} x_{10}(1-t): & t \leq 1, \\ 0: & t > 1, \end{cases}$$

and (24) yields that $x_3(t) = \frac{\mathrm{d}}{\mathrm{d}t}x_2(t)$*. This requires continuous differentiability of* x_2*, which is fulfilled, if* $x_0 \in H^3([0,1])$ *with* $x_0(0) = x_0'(0) = x_0''(0) = 0$*. In this case,*

$$x_3(t) = \begin{cases} -\frac{\partial}{\partial\xi}x_{10}(1-t): & t \leq 1, \\ 0: & t > 1. \end{cases}$$

Hence, a classical solution of (21)–(24) exists, if

$$x_{10} \in H^3([0,1]) \quad \textit{with} \quad x_{10}(0) = x_{10}'(0) = x_{10}''(0) = 0.$$

Next we analyse the differentiation index $p_{\mathrm{diff}}^{(E,A)}$*. First, by noticing that* E *has a nontrivial nullspace, we must have that* $p_{\mathrm{diff}}^{(E,A)} > 0$*.*
On the other hand, since, for

$$x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in X,$$

it holds that

$$\begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} \in \ker M_1$$

one has $p_{\mathrm{diff}}^{(E,A)} > 1$*. It will now be shown that* $p_{\mathrm{diff}}^{(E,A)} = 2$*. To this end, let* $x_k \in X$*,* $k = 0,1,2,3$*, partitioned as*

$$x_k = \begin{pmatrix} x_{k1} \\ x_{k2} \\ x_{k3} \end{pmatrix}, \quad x_{k1} \in L^2(0,1),\, x_{k2}, x_{k3} \in \mathbb{C},$$

and

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \ker M_2. \tag{33}$$

Then

$$\tfrac{\partial}{\partial\xi}x_{11} = -x_{21}, \quad \tfrac{\partial}{\partial\xi}x_{21} = -x_{21},$$

and

$$0 = \delta_0 x_{11}, \quad 0 = \delta_0 x_{21} = -\delta_0 \tfrac{\partial}{\partial\xi}x_{11},$$

which implies that

$$x_{11} \in \tilde{X}_1 := \left\{ x \in H^2(0,1) \;\middle|\; x(0) = x'(0) = 0 \right\}.$$

Analogously, we see that

$$x_{01} \in \tilde{Y}_1 := \left\{ x \in H^3(0,1) \;\middle|\; x(0) = x'(0) = x''(0) = 0 \right\}.$$

Further, by

$$\begin{aligned} x_{11} &= -\tfrac{\partial}{\partial\xi}x_{10}, \\ x_{12} &= x_{03}, \\ x_{13} &= x_{22} = \delta_1 x_{21} = -\delta_1 \tfrac{\partial}{\partial\xi}x_{11} = \delta_1 \tfrac{\partial^2}{\partial\xi^2}x_{10}, \end{aligned}$$

we have that (33) implies that, for

$$\tilde{X} := \tilde{X}_1 \times \mathbb{C}^2$$

and $(S, \mathrm{dom}(S))$ *with*

$$\mathrm{dom}(S) = \tilde{Y}_1 \times \mathbb{C}^2, \quad S\begin{pmatrix} x_{01} \\ x_{02} \\ x_{03} \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial\xi}x_{01} \\ x_{03} \\ x_{10}''(1) \end{pmatrix}.$$

The operator S is indeed the generator of a strongly continuous semigroup on $\tilde{X}$, namely $T(\cdot)$ with

$$T(t)\begin{pmatrix}x_{01}\\x_{02}\\x_{03}\end{pmatrix}=\begin{pmatrix}\mathbb{S}_t x_{01}\\x_{02}+tx_{03}+\int_0^t\int_0^\tau\left(\mathbb{S}_s x_{10}\right)(1)\,\mathrm{d}s\,\mathrm{d}\tau\\x_{03}+\int_0^t\left(\mathbb{S}_\tau x_{10}\right)(1)\,\mathrm{d}\tau\end{pmatrix},$$

where $\mathbb{S}_t$ denotes the right shift of length t on functions defined on the interval $[0,1]$.
In the case where the initial condition $x(0)=x_0=(x_{01},x_{02},x_{03})$ for the DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t)=Ax(t)$ fulfills

$$x_{01}\in X_1,\quad x_{02}=x_{01}(1),\quad x_{03}=-x_{01}'(1),$$

it can be seen that

$$x(t)=T(t)\begin{pmatrix}x_{01}\\x_{02}\\x_{03}\end{pmatrix}$$

is indeed a solution of (21)-(24).

Proposition 7.6. *(i) If the differentiation index $p_{\mathrm{diff}}^{(E,A)}$ exists, then the chain index exists and $p_{\mathrm{chain}}^{(E,A)}\leq p_{\mathrm{diff}}^{(E,A)}$.*

(ii) If the nilpotency index $p_{\mathrm{nilp}}^{(E,A)}$ exists and the operator A_1 in the Weierstraß form generates a C_0-semigroup, then the differentiation index exists and $p_{\mathrm{nilp}}^{(E,A)}=p_{\mathrm{diff}}^{(E,A)}$.

Proof. (i) Assume that the differentiation index $p_{\mathrm{diff}}^{(E,A)}$ exists. Further, let $(x_1,\ldots,x_p)$, $p\in\mathbb{N}$, be a chain of (E,A). Then

$$M_{p-1}\begin{pmatrix}0\\x_1\\\vdots\\x_p\end{pmatrix}=0.$$

This implies that there does not exist a mapping $S\colon\operatorname{dom}(S)\to X$ such that every $z\in\ker M_{p-1}$ satisfies

$$z=\begin{pmatrix}x_0\\Sx_0\\x_2\\\vdots\\x_p\end{pmatrix}$$

for some $x_0\in X$, $x_2,\ldots,x_p\in\operatorname{dom}(A)$. Consequently, $p\leq p_{\mathrm{diff}}^{(E,A)}$, that is, the chain index of (E,A) does not exceed the differentiation index of (E,A).

(ii) Assume that (E,A) is in Weierstraß form (2) with $\nu=p_{\mathrm{nilp}}^{(E,A)}$ and $N^\nu=0$, $N^{\nu-1}\neq0$. Let $x_k\in X=Y^1\times Y^2$, $k\in\mathbb{N}$, be partitioned as

$$x_k=\left(\begin{smallmatrix}x_{k,1}\\x_{k,2}\end{smallmatrix}\right),\quad x_{k,1}\in Y^1,\ x_{k,2}\in Y^2.$$

Using the Weierstraß form, it follows that

$$\begin{pmatrix}x_0\\\vdots\\x_{k+1}\end{pmatrix}\in\ker M_k \tag{34}$$

if, and only if, for some $x_{1,0}\in\operatorname{dom}(A_1^{k+1})$, $x_{2,0}\in Y^2$,

$$x_{1,i}=A_1^i x_{1,0},\quad N^i x_{2,i}=x_{2,0},\quad i=1,\ldots,k+1. \tag{35}$$

Let $k<\nu$. Then by choosing some $z\in Y^2$ with $N^{\nu-1}z\neq0$, we obtain (34) with

$$x_0=\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),x_1=\left(\begin{smallmatrix}0\\N^{\nu-1}z\end{smallmatrix}\right),\ldots,x_{k+1}=\left(\begin{smallmatrix}0\\N^{\nu-k-1}z\end{smallmatrix}\right).$$

Since $x_0 = 0$ and $x_1 \neq 0$, an operator S with the properties as described in Definition 7.1 cannot exist.

On the other hand, the equivalence between (34) and (35) implies that if $k = \nu$

$$x_1 = \begin{pmatrix} A_1 x_{0,1} \\ 0 \end{pmatrix}.$$

Consequently, we can choose $\tilde{X} = X$ and $S\colon\ \mathrm{dom}(S) \subset X \to X$ with $\mathrm{dom}(S) = \mathrm{dom}(A_1) \times Y^2$ and

$$S = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is the generator of the strongly continuous semigroup

$$T(\cdot) = \begin{bmatrix} T_1(\cdot) & 0 \\ 0 & I_{Y^2} \end{bmatrix},$$

where $T_1(\cdot)$ is the semigroup generated by A_1. This shows that the differentiation index coincides with the nilpotency index of N. $\square$

8 Perturbation index

Finally, we define the perturbation index for DAEs on possibly infinite dimensional spaces. As the name indicates, the perturbation index is a measure of sensitivity of solutions with respect to perturbations of the problem.

Definition 8.1 (perturbation index).
Consider the DAE (1) on $[0,T]$ with input $\delta \in C^p([0,T],Z)$ and solution x:

$$\frac{d}{dt}Ex(t) = Ax(t) + \delta(t), \quad t \geq 0. \tag{36}$$

The perturbation index $p_{\mathrm{pert}}^{(E,A)}$ is the smallest number $p \in \mathbb{N}$ such that there exist a constant $c > 0$ satisfying the following for all δ:

$$\max_{0\le t\le T} \|x(t)\|_X \leq c\left(\|x(0)\|_X + \sum_{i=0}^{p-1} \max_{0\le t\le T}\left\|\frac{d^i}{dt^i}\delta(t)\right\|_Z\right), \qquad \text{if } p > 0, \tag{37}$$

$$\max_{0\le t\le T} \|x(t)\|_X \leq c\left(\|x(0)\|_X + \int_0^T \|\delta(t)\|_Z \,\mathrm{d}t\right), \qquad \text{if } p = 0. \tag{38}$$

Notice that this definition differs from the definition of the perturbation index for infinite dimensional systems given in [5]. Here, we measure the influence of perturbations with respect to time only. In case of constrained PDE systems, also derivatives with respect to space can be involved but they are hidden in the norm $\|\cdot\|_Z$ and are not counted.

First we show that the perturbation index is identical for equivalent systems.

Proposition 8.2. *The perturbation index, if it exists, is uniquely defined. To be more precise, let $(E,A) \sim (\tilde{E},\tilde{A})$. Then $p_{\mathrm{pert}}^{(E,A)} = p_{\mathrm{pert}}^{(\tilde{E},\tilde{A})}$.*

Proof. Let $P\colon X \to \tilde{X}$, $Q\colon Z \to \tilde{Z}$, be bounded isomorphisms as in Definition 2.1. Then the result follows, since x is a solution of (36), if, and only if, $\tilde{x} = Px$ is a solution of $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}\tilde{x} = \tilde{A}\tilde{x} + \tilde{\delta}$ for $\tilde{\delta} = Q^{-1}\delta$. $\square$

Proposition 8.3. *The following holds:*

(i) If the nilpotency index $p_{\mathrm{nilp}}^{(E,A)}$ *and the perturbation index* $p_{\mathrm{pert}}^{(E,A)}$ *exist, then* $p_{\mathrm{nilp}}^{(E,A)} \leq p_{\mathrm{pert}}^{(E,A)}$.

(ii) If the nilpotency index $p_{\mathrm{nilp}}^{(E,A)}$ *exists and* A_1 *in (2) generates a* C_0*-semigroup, then the perturbation index also exists and* $p_{\mathrm{nilp}}^{(E,A)} = p_{\mathrm{pert}}^{(E,A)}$.

Proof. By Proposition 8.2, it is no loss of generality to assume that the system is in Weierstraß form, that is,

$$(E,A) = \left(\begin{bmatrix} I_{Y^1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{Y^2} \end{bmatrix} \right).$$

First, we show Part (i). If $p_{\mathrm{nilp}}^{(E,A)} = 0$, then nothing has to be shown. If $p_{\mathrm{nilp}}^{(E,A)} = 1$, then $N = 0$ and thus, for the second part of a solution x of (36) holds $x_2(t) = -\delta_2(t)$. Consequently, the maximum of x cannot be estimated by the integral of $\|\delta_2\|$, and thus also an estimate of the form (38) is not possible. Thus, we have $p_{\mathrm{pert}}^{(E,A)} \leq 1$ in this case.
Finally, we assume that $p := p_{\mathrm{nilp}}^{(E,A)} \geq 2$. Seeking for a contradiction, we assume that $p_{\mathrm{pert}}^{(E,A)} < p$. Then there exist some $c > 0$, such that for all solutions x of (36), it holds

$$\max_{0 \leq t \leq T} \|x(t)\|_Y \leq c \left(\|x(0)\|_Y + \sum_{i=0}^{p-2} \max_{0 \leq t \leq T} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i} \delta(t) \right\|_Y \right). \tag{39}$$

Let $v_2 \in Y_2$ with $N^{p-1} v_2 \neq 0$. We choose $\delta_1 = 0$ $x_1(0) = 0$ and $\delta_{2,n} \colon [0,T] \to Y^2$ with

$$\delta_{2,n}(t) = \frac{\mathrm{Re}\left(\imath^p \, \mathrm{e}^{\imath nt}\right)}{n^{p-1}} v_2.$$

Then, by using that $\mathrm{Re}\left(\imath^p \, \mathrm{e}^{\imath nt}\right)$ is a trigonometric function, we have

$$\forall n \geq \frac{\pi}{T} : \quad \max_{0 \leq t \leq T} \|N^{p-1} \delta_{2,n}^{(p-1)}\|_{Y^2} \geq \|N^{p-1} v_2\|_{Y^2} > 0, \tag{40}$$

and, further

$$\forall k = 1, \ldots, p-2 : \quad \lim_{n \to \infty} \max_{0 \leq t \leq T} \|\delta_{2,n}^{(k)}\|_{Y^2} = 0, \tag{41}$$

$$\delta_{2,n}(0) = \mathrm{Re}\,\left(\imath \, \mathrm{e}^{\imath nt}\right)\Big|_{t=0} v_2 = 0. \tag{42}$$

Let $\delta_n = \left(\begin{smallmatrix} 0 \\ \delta_{2,n} \end{smallmatrix}\right)$. Since if $x_{1,n} = 0$ the solution $x_n = \left(\begin{smallmatrix} x_{1,n} \\ x_{2,n} \end{smallmatrix}\right)$ of $\frac{\mathrm{d}}{\mathrm{d}t} E x_n(t) = A x_n(t) + \delta_n(t)$ has $x_{1,n}(0) = 0$ and also

$$x_{2,n} = -\sum_{i=0}^{p-1} N^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \delta_{2,n}(t),$$

we obtain from (40) and (41) that

$$\liminf_{n \to \infty} \max_{0 \leq t \leq T} \|x_n(t)\|_Y > 0.$$

On the other hand, by (40) and (42), we obtain

$$\lim_{n \to \infty} \underbrace{\|x_n(0)\|_Y}_{=0} + \sum_{i=0}^{p-2} \max_{0 \leq t \leq T} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i} \delta_n(t) \right\|_Y = 0,$$

which contradicts (39).

Next, we will show Part (ii). Let x be a solution of DAE (36), and partition

$$x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix},\quad \delta(t)=\begin{pmatrix}\delta_1(t)\\\delta_2(t)\end{pmatrix}$$

according to the structure of the Weierstraß form. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t))=A_1x_1(t)+\delta_1(t),\quad t\geq 0, \tag{43}$$

$$N\frac{\mathrm{d}}{\mathrm{d}t}x_2(t)=x_2(t)+\delta_2(t),\quad t\geq 0. \tag{44}$$

Let S be the semigroup generated by A_1. Then (43) yields

$$x_1(t)=S(t)x_1(0)+\int_0^t S(t-\tau)\delta_1(\tau)\,\mathrm{d}\tau.$$

Since S is bounded on $[0,T]$, we have $c_0:=\sup_{t\in[0,T]}\|S(t)\|<\infty$, $c_1=\max\{c_0,c_0T\}$, and we can conclude

$$\begin{aligned}\max_{t\in[0,T]}\|x_1(t)\|_{Y^1}&\leq c_0\|x_1(0)\|_{Y^1}+Tc_0\max_{0\leq t\leq T}\|\delta_1(t)\|_{Y^1} &(45)\\ &\leq c_1\big(\|x_1(0)\|_{Y^1}+\max_{0\leq t\leq T}\|\delta_1(t)\|_{Y^1}\big). &(46)\end{aligned}$$

In the case where $p=0$, we have $X=Y^1$, $x=x_1$ and $\delta=\delta_1$, and we obtain from (45) that $p_{\text{pert}}^{(E,A)}=0$. If this is not the case, we find from (44) that

$$\begin{aligned}x_2(t)&=N\frac{\mathrm{d}}{\mathrm{d}t}x_2(t)-\delta_2(t)=N\frac{\mathrm{d}}{\mathrm{d}t}\left(N\frac{\mathrm{d}}{\mathrm{d}t}x_2(t)-\delta_2(t)\right)-\delta_2(t)\\ &=N^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_2(t)-N\frac{\mathrm{d}}{\mathrm{d}t}\delta_2(t)-\delta_2(t)=N^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(N\frac{\mathrm{d}}{\mathrm{d}t}x_2(t)-\delta_2(t)\right)-N\frac{\mathrm{d}}{\mathrm{d}t}\delta_2(t)-\delta_2(t)\\ &=N^3\frac{\mathrm{d}^3}{\mathrm{d}t^3}x_2(t)-N^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta_2(t)-N\frac{\mathrm{d}}{\mathrm{d}t}\delta_2(t)-\delta_2(t)=\ldots=-\sum_{i=0}^{p-1}N^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\delta_2(t).\end{aligned}$$

Boundedness of E yields that N is bounded, and thus $c_2:=\max_{i=0}^{p-1}\|N^i\|<\infty$. Consequently,

$$\max_{0\leq t\leq T}\|x_2(t)\|_{Y^2}\leq c_2\left(\sum_{i=0}^{p-1}\max_{0\leq t\leq T}\left\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\delta_2(t)\right\|_{Y^2}\right). \tag{47}$$

Collecting the bounds (45) and (47) implies that, setting $c:=\max\{c_1,Tc_1,c_2\}$,

$$\max_{0\leq t\leq T}\|x(t)\|_Y\leq c\left(\|x(0)\|_Y+\sum_{i=0}^{p-1}\max_{0\leq t\leq T}\left\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\delta(t)\right\|_Y\right).$$

Thus, $p_{\text{pert}}^{(E,A)}\leq p=p_{\text{nilp}}^{(E,A)}$. □

Remark 8.4. *In [29], a class of linear infinite dimensional differential-algebraic systems was examined, which do not necessarily have a nilpotency index. It was revealed that a perturbation analysis necessitates the consideration of stronger norms for the initial value. These norms may correspond to a type of spatial perturbation index for systems governed by partial differential-algebraic equations.*

9 Conclusions

The purpose of this paper was, as explained in the introduction, to review the common indices for DAEs and to extend them to infinite dimensional DAEs where this has not already been done. Rigorous definitions were given for the nilpotency, resolvent, radiality, chain, differentiation and perturbation indices. The definitions of the differentiation and perturbation indices are as far as we know, new in the Banach space setting. Unlike finite dimensional DAEs, the existence of an index is not guaranteed, and furthermore the various indices are not equivalent for general DAEs. Each index reveals different information about the system. This is illustrated by providing at least one example for each index, and highlighting situations where one index exists but not another.

We showed that in some cases existence of a particular index will imply existence of some other other indices. One result is that the chain index is the least restrictive index since existence of any of the other indices implies existence of the chain index, as well as a bound on the chain index. Propositions 5.4 and 6.3 imply that if the radiality and nilpotent indices exist, then the resolvent and chain indices exist and

$$p_{\text{rad}}^{(E,A)} + 1 \geq p_{\text{res}}^{(E,A)} \geq p_{\text{nilp}}^{(E,A)} = p_{\text{chain}}^{(E,A)}.$$

If in addition, A_1 in the Weierstraß form generates a C_0-semigroup, then all the other indices exist (Prop. 7.6 and 8.3) and

$$p_{\text{rad}}^{(E,A)} + 1 \geq p_{\text{res}}^{(E,A)} \geq p_{\text{nilp}}^{(E,A)} = p_{\text{diff}}^{(E,A)} = p_{\text{chain}}^{(E,A)} = p_{\text{pert}}^{(E,A)}.$$

If the underlying spaces are finite dimensional, then all the indices exist and are equivalent (Prop. 6.9).

This work has revealed a number of open questions for DAEs on Banach spaces. Under what conditions does a particular index exist? If two indices exist, when are they equal? Which indices are more useful for the study of DAEs? How does the index change under discretizations? It is hoped that this paper will inspire research on these and related questions.

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No new data were created or analysed during this study. Data sharing is not applicable to this article.

Underlying and related material

No underlying or related material.

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All authors contributed equally.

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No competing interests to declare.

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Mehmet Erbay
IMACM
University of Wuppertal
Gaußstraße 20
42119 Wuppertal, Germany
erbay@uni-wuppertal.de

Birgit Jacob
IMACM
University of Wuppertal
Gaußstraße 20
42119 Wuppertal, Germany
bjacob@uni-wuppertal.de

Kirsten Morris
Departmend of Applied Mathematics
University of Waterloo
200 University Avenue West
Waterloo On N2l 3G1, Canada
kmorris@uwaterloo.ca

Timo Reis
Institut für Mathematik
Technische Universität Ilmenau
Weimarer Str. 25
98693 Ilmenau, Germany
timo.reis@tu-ilmenau.de

Caren Tischendorf
Lehrstuhl für Angewandte Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
tischendorf@math.hu-berlin.de