



Canonical Subspaces of Linear Time-Varying Differential-Algebraic Equations and Their Usefulness for Formulating Accurate Initial Conditions

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Abstract: Accurate initial conditions have the task of precisely capturing and fixing the free integration constants of the flow considered. This is trivial for regular ordinary differential equations, but a complex problem for differential-algebraic equations (DAEs) because, for the latter, these free constants are hidden in the flow. We deal with linear time-varying DAEs and obtain an accurate initial condition by means of applying both a reduction technique and a projector based analysis. The highlighting of two canonical subspaces, the flow-subspace and its canonical complement, plays a special role. In order to be able to apply different DAE concepts simultaneously, we first show that the very different looking rank conditions on which the regularity notions of the different concepts (elimination of unknowns, reduction, dissection, strangeness, and tractability) are based are de facto consistent. This allows an understanding of regularity independent of the methods.

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1. Introduction

In the context of generalized eigenvalue problems, regular matrix pencils, and descriptor systems, so-called spectral projectors and their associated subspaces receive special attention. In contrast, they have received little attention in the context of the analysis of differential-algebraic equations (DAEs). We dedicate the present work to this question.

In doing so, we take as a basis the *regularity notion* that adopts the reduction idea presented in [25]. We will prove that this notion captures equally proper the very different approaches from [5, 16, 25, 17, 18]. In particular, we provide a set of characteristic values from which one can derive the characteristic values of the different approaches, and which, in this sense, are method-independent.

We deal both with DAEs in standard form,

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \quad (1)$$

and DAEs with properly involved derivative term,

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \quad (2)$$

with sufficiently smooth coefficient functions $E, F : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$, and $A : \mathcal{I} \rightarrow \mathbb{R}^{m \times k}$, $D : \mathcal{I} \rightarrow \mathbb{R}^{k \times m}$, $B : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$, respectively.

The properly involved derivative is merely useful when it comes to exact smoothness issues of the solutions. Such properties do not matter here, since we assume continuously differentiable solutions, as is widely customary and even necessary for the use of the related concepts. Here we are interested in two particular time-varying subspaces of \mathbb{R}^m where solution values are located. The first canonical subspace $S_{can}(t)$ is defined as the set of all solution values of the homogeneous DAE at time t . We call it *flow-subspace* of the DAE. The second canonical subspace $N_{can}(t)$ is a particular complement to $S_{can}(t)$ to be further specified later on such that

$$S_{can}(t) \oplus N_{can}(t) = \mathbb{R}^m.$$

We will call $N_{can}(t)$ the *canonical complement to the flow-subspace*. As a matter of course, the canonical subspaces S_{can} , N_{can} , as well as the associated projector function Π_{can} should not depend on how the leading term of the DAE is written down.

We will prove that any regular DAE (1) or (2) with sufficiently smooth coefficients possesses a well-defined canonical projector function Π_{can} with associated subspaces S_{can} and N_{can} in analogy to, and as a generalization of, the spectral projector. Then the reduction procedure from [25] provides a basis of the canonical subspace S_{can} .

The knowledge of the subspace N_{can} allows to elaborate *accurately stated initial conditions* in the sense of [19, Definition 2.3] and [18, Theorem 2.52], i.e., the accurate capture of the implicit free integration constants. Using a basis of the corresponding canonical subspace S_{*can} to the adjoint DAE, we obtain a suitable matrix which allows to formulate accurate initial conditions.

Accurately stated initial conditions are something quite different from consistent initial values. The latter are needed in integration codes so far, but the generation of consistent initial data is a problem on its own, e.g., [6, 3, 2].

The least-squares collocation procedures for initial and boundary value problems with higher index differential-algebraic equations proposed and analyzed in [12, 11, 8, 9, 14, 15, 13] work incredibly good, but one has to provide accurate initial and boundary conditions. Moreover, for the time-stepping version [10] for systems with dynamic degrees of freedom, one needs also accurately stated transition conditions from one window to the next.

The paper is structured as follows: In Section 2, as an introduction, we recapitulate the state of affairs for matrix pencils and DAEs with constant coefficients as well as for semi-explicit index-one DAEs. Then, in Section 3, we use the reduction approach of [25], agree on the regularity notion, and provide the first canonical subspace S_{can} , which is related to the flow of the homogeneous DAE. The following Section 4 is concerned with a comparison of different concepts, which in the end will show the general validity of the agreed notion of regularity together with its characteristics. This allows the simultaneous use of the results of the different associated theories. In Section 5 we elaborate what is meant by accurate initial conditions and the role of the second canonical subspace N_{can} . The practically most important result is then stated in Theorem 6.2 in Section 6. In this theorem, we derive a matrix, suitable for formulating accurate initial conditions, by using information coming from the adjoint DAE. Further, a nontrivial example is discussed to some extent in Section 7. There is an appendix with additional information about some technical details.

2. A review of two well-known cases for motivation, and orientation

2.1. A look back at the well-understood regular matrix pencils

Given the ordered pair $\{E, F\}$ of matrices $E, F \in \mathbb{R}^{m \times m}$, we consider the expression $p(\lambda) = \det(\lambda E + F)$ being a polynomial in λ , and the so-called matrix pencil $\lambda E + F$. If the polynomial does not vanish identically, the matrix pencil is said to be *regular*. In turn, the pair $\{E, F\}$ is said to be *regular* then.

For each regular pair $\{E, F\}$, there exist two integers $d, \mu \geq 0$, $\mu \leq m - d$, and nonsingular matrices $L, K \in \mathbb{R}^{m \times m}$ such that

$$LEK = \begin{bmatrix} I_d & \\ & \mathcal{N} \end{bmatrix}, \quad LFK = \begin{bmatrix} \mathcal{W} & \\ & I_{m-d} \end{bmatrix}. \quad (3)$$

Here and in the following, $I_k \in \mathbb{R}^{k \times k}$ denotes the identity matrix for $k \in \mathbb{N}$. Thereby, \mathcal{N} is absent if $d = m$. Otherwise, \mathcal{N} is nilpotent with index μ , i.e., $\mathcal{N}^\mu = 0$, $\mathcal{N}^{\mu-1} \neq 0$. Both integers μ, d , the sizes of the blocks \mathcal{N} and \mathcal{W} , their eigenstructure as well as the subspaces

$$S_{can} := \text{im} K \begin{bmatrix} I_d \\ 0 \end{bmatrix} \subset \mathbb{R}^m, \quad N_{can} := \text{im} K \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix} \subset \mathbb{R}^m, \quad (4)$$

are uniquely determined by the matrix pencil or the pair $\{E, F\}$. Formula (3) describes the *Weierstraß–Kronecker form*¹ of the matrix pencil and pair, respectively, and μ is the *Kronecker index*. We refer to [18, Proposition 1.3] for corresponding proofs, because there the proof is arranged in such a way that also the less often recognized fact (4) results immediately². The *canonical subspaces* S_{can} and N_{can} associated to the pencil $\lambda E + F$ or pair $\{E, F\}$ have dimensions d and $m - d$, respectively. They intersect trivially only such that the decomposition

$$S_{can} \oplus N_{can} = \mathbb{R}^m$$

becomes true. The so-called *canonical or spectral projector*³ Π_{can} onto S_{can} along N_{can} turns out to be also uniquely determined by $\{E, F\}$ and a possible representation reads

$$\Pi_{can} = K \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} K^{-1}.$$

At this place it should be noted that, in the linear algebra context, for a regular pair $\{E, F\}$ with $d < m$, so-called complementary deflating subspaces corresponding to its finite and infinite eigenvalues play their role. Π_{can} is then the (right) spectral projector onto the right deflating subspace and $L^{-1} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} L$ is the (left) spectral projector onto the left deflating subspace. The latter hardly plays a role in DAE theory, which is why the addition "right" is omitted here for the spectral projector Π_{can} .

It is worth adding that always $\ker E \subseteq N_{can}$ holds true.

The linear constant coefficient DAE associated with the regular pair $\{E, F\}$,

$$Ex'(t) + Fx(t) = q(t), \quad t \in \mathcal{I}, \quad (5)$$

¹In [1, 27], the term *Quasi-Weierstraß* is used to emphasize that, unlike the standard Kronecker form, no Jordan form is required for \mathcal{N} and \mathcal{W} here.

²In terms of the proof in [18, Page 6]: Using $\tilde{K}^{-1}K = \text{diag}(V_{11}, V_{22})$ it simply results that

$$\begin{aligned} \text{im} K \begin{bmatrix} I_d \\ 0 \end{bmatrix} &= \text{im} \tilde{K} \tilde{K}^{-1} K \begin{bmatrix} I_d \\ 0 \end{bmatrix} = \text{im} \tilde{K} \begin{bmatrix} I_d \\ 0 \end{bmatrix} V_{11} = \text{im} \tilde{K} \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \\ \text{im} K \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix} &= \text{im} \tilde{K} \tilde{K}^{-1} K \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix} = \text{im} \tilde{K} \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix} V_{22} = \text{im} \tilde{K} \begin{bmatrix} 0 \\ I_{m-d} \end{bmatrix}. \end{aligned}$$

³Also *consistency projector* in [27].

decomposes correspondingly into

$$\begin{aligned} u'(t) + \mathcal{W}u(t) &= f(t), \\ \mathcal{N}v'(t) + v(t) &= g(t), \quad t \in \mathcal{I}, \end{aligned}$$

with

$$Lq(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \quad x(t) = K \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \underbrace{K \begin{bmatrix} u(t) \\ 0 \end{bmatrix}}_{\in S_{can}} + \underbrace{K \begin{bmatrix} 0 \\ v(t) \end{bmatrix}}_{\in N_{can}}.$$

Each solution of the DAE (5) has the form

$$x(t) = K \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad (6)$$

$$u(t) = \exp^{-(t-a)\mathcal{W}} u(a) + \int_a^t \exp^{(s-t)\mathcal{W}} f(s) ds,$$

$$v(t) = \sum_{j=0}^{\mu-1} (-1)^j \mathcal{N}^j g^{(j)}(t), \quad t \in \mathcal{I}, \quad a \in \mathcal{I}. \quad (7)$$

Clearly, the integer d is at the same time the dynamical degree of freedom of the system (5). The case $d = 0$, and hence $\Pi_{can} = 0$, $S_{can} = \{0\}$ may happen.

At this place, we emphasize that, when dealing with DAEs, one has to assume that neither the transformation matrices K, L nor the canonical subspaces together with spectral projectors are practically available with little effort. Calculating them is considerably easier⁴ than solving general eigenvalue problems, but still quite complex and rather hopeless for the case of variable coefficients, which we are actually interested in.

In case of the homogeneous DAE (5) with $d > 0$, the flow is completely restricted to remain in the subspace S_{can} and in turn, at each time $a \in \mathcal{I}$, and arbitrary $x_a \in S_{can}$, there is exactly one solution passing through, i.e., $x(a) = x_a$. The situation is quite different for nonhomogeneous DAEs. Only if the function g vanishes identically,⁵ and in turn $v(t) \equiv 0$, then the flow is again restricted to S_{can} . The property $g(t) \equiv 0$ is practically very unlikely and unrecognizable. Otherwise, one is confronted with nontrivial parts $v(t)$ in (6). In particular, for given $a \in \mathcal{I}$, $x_a \in S_{can}$, it results that

$$x_a = K \begin{bmatrix} u_a \\ 0 \end{bmatrix}, \quad x(a) - x_a = \underbrace{K \begin{bmatrix} u(a) - u_a \\ 0 \end{bmatrix}}_{\in S_{can}} + \underbrace{K \begin{bmatrix} 0 \\ v(a) \end{bmatrix}}_{\in N_{can}},$$

and, obviously, the conditions

$$x(a) - x_a \in N_{can}, \quad (8)$$

$$\text{or, equivalently, } \Pi_{can} x(a) = x_a,$$

imply $u(a) = u_a$, and vice versa.

Also for $x_a \in \mathbb{R}^m$ (instead of $x_a \in S_{can}$) we have

$$x(a) - x_a = \underbrace{K \begin{bmatrix} u(a) - u_a \\ 0 \end{bmatrix}}_{\in S_{can}} + \underbrace{K \begin{bmatrix} 0 \\ v(a) - v_a \end{bmatrix}}_{\in N_{can}},$$

⁴Here *Wong sequences* [1, 27, 28] offer themselves. Furthermore, [18, Proposition 1.3] is constructive and the canonical projector can be determined along the line of [22, 29] and [18, Part III].

⁵This means, that all $q(t)$ are completely restricted to remain within the left deflating subspace.

and the conditions

$$x(a) - x_a \in N_{can}, \quad (9)$$

$$\text{or, equivalently, } \Pi_{can}x(a) = \Pi_{can}x_a, \quad (10)$$

imply $u(a) = u_a$, and vice versa. We emphasize the unique role of the subspace N_{can} in the DAE context shown here. Only the knowledge of this subspace allows an accurate fixation of the corresponding single solution with property (10), and this even without knowledge of the projector Π_{can} itself.

Since each function value $x(a)$ may incorporate a nontrivial $v(a)$, the initial condition $x(a) = x_a \in \mathbb{R}^m$ includes the necessary condition $v(a) = v_a$ which means that the corresponding differentiations in (7) for providing $v(a)$ must be carried out. To really calculate such a kind of *consistent initial values* is quite laborious and uncertain. However, if x_a is not consistent, then the initial value problem (IVP) is not solvable.

On the other hand, if one assumes that the initial conditions are to fix a unique solution from the flow, then exactly d initial conditions shall be set. The following three equivalent conditions are useful,

$$\Pi(x(a) - x_a) = 0, \quad (11)$$

$$x(a) - x_a \in N_{can}, \quad (12)$$

$$G_a(x(a) - x_a) = 0, \quad (13)$$

in which the matrix $G_a \in \mathbb{R}^{d \times m}$ has full row-rank d and its nullspace coincides with N_{can} , that is, $\ker G_a = N_{can}$, and $\Pi \in \mathbb{R}^{m \times m}$ is any projector matrix such that $\ker \Pi = N_{can}$, among others Π_{can} is allowed. Also, for any sufficiently smooth inhomogeneity q , an IVP with any of these conditions is uniquely solvable without any further consistency conditions related to q having to be satisfied. In this sense, these initial conditions are accurately formulated.

We conclude our brief review by showing that various fundamental matrices can be defined for the DAE, all of which are now pointwise singular, have constant rank d , and whose pointwise image is exactly S_{can} . Typical here are on the one hand the maximal-size version $X(t)$ using the canonical projector, and on the other hand the minimal-size version $X_b(t)$ using a basis C of S_{can} given by

$$\begin{aligned} EX'(t) + FX(t) &= 0, & X(a) &= \Pi_{can}, \\ EX'_b(t) + FX_b(t) &= 0, & X_b(a) &= C. \end{aligned}$$

Both versions have their justification. The canonical projector contains more information, namely that about both N_{can} and S_{can} , and the corresponding maximal fundamental solution has semigroup properties analogous to the ODE case, e.g., [18], which is very helpful for analytical studies. On the other hand, it is usually easier to compute a basis numerically than a projector, especially if it is not an orthoprojector and thus one needs, say, the bases of two subspaces.

Among other things, we also show below a new way to compute the canonical projector Π_{can} which seems to be more convenient than the iterative original version in [22]. In the present paper, by means of the reduction technique presented in Section 3, a basis C of S_{can} and a basis C_* of the subspace S_{*can} associated with the adjoint matrix pair $\{-E^*, F^*\}$ are determined. From the relations

$$\text{im } C = S_{can}, \quad \ker C_*^* E = N_{can}$$

which are valid according to Theorem 6.2, one can then derive the projector by usual numerical algebra techniques. Moreover, we may choose $G_a = C_*^* E$ for IVPs.

2.2. Semi-explicit index-one DAEs

Let the matrix function $F : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ be sufficiently smooth. The pair given by

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

is associated with the DAE $Ex' + Fx = q$; in more detail,

$$\begin{aligned} x_1'(t) + F_{11}(t)x_1(t) + F_{12}(t)x_2(t) &= q_1(t), \\ F_{21}(t)x_1(t) + F_{22}(t)x_2(t) &= q_2(t). \end{aligned}$$

Let the entry $F_{22}(t) \in \mathbb{R}^{(m-r) \times (m-r)}$ remain nonsingular on \mathcal{I} . Then the DAE is regular with index one and possesses the basic subspaces

$$\begin{aligned} N &= \ker E = \text{im} \begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix}, \\ S(t) &= \{z \in \mathbb{R}^m : F(t)z \in \text{im} E\} = \text{im} C(t), \quad C(t) = \begin{bmatrix} I_r \\ -F_{22}(t)^{-1}F_{21}(t) \end{bmatrix}, \\ N \cap S(t) &= \{0\}, \quad t \in \mathcal{I}. \end{aligned} \tag{14}$$

such that $\mathbb{R}^m = S(t) \oplus N$.

For better clarity, we usually omit the t -argument in the following. The relations are then meant point-wise.

Each solution $x : \mathcal{I} \rightarrow \mathbb{R}^m$ of the DAE has the form

$$x = \begin{bmatrix} Ux_1(a) + f \\ -F_{22}^{-1}F_{21}(Ux_1(a) + f) + F_{22}^{-1}q_2 \end{bmatrix} = C(Ux_1(a) + f) + \begin{bmatrix} 0 \\ F_{22}^{-1}q_2 \end{bmatrix}, \tag{15}$$

in which $U : \mathcal{I} \rightarrow \mathbb{R}^{r \times r}$ denotes the solution of

$$U' + (F_{11} - F_{12}F_{22}^{-1}F_{21})U = 0, \quad U(a) = I_r,$$

$x_1(a)$ can be predefined arbitrarily by initial conditions, and

$$f(t) = U(t) \int_a^t U(s)^{-1} (q_1(s) - F_{12}(s)F_{22}(s)^{-1}q_2(s)) ds, \quad t \in \mathcal{I}.$$

Hereby $d = r$ is the dynamical degree of freedom. Further, if $q = 0$, in turn $f = 0$, the solutions simplify to

$$x = \begin{bmatrix} Ux_1(a) \\ -F_{22}^{-1}F_{21}Ux_1(a) \end{bmatrix} = CUx_1(a), \quad x(a) = C(a)x_1(a).$$

Since $C(a)$ is a basis of $S(a)$, and if $S_{can}(a)$ is to be again the subspace containing all solution values of the homogeneous DAE at time a , it results that $S = S_{can}$.

Next we are looking for an appropriate complementary to S_{can} subspace N_{can} in view of accurately stated initial conditions. The sought subspace must have dimension $m - r$.

Let $x_a = \begin{bmatrix} x_{a1} \\ x_{a2} \end{bmatrix} \in \mathbb{R}^m$ with $x_{a1} \in \mathbb{R}^r$. Obviously, the condition $x(a) - x_a \in N$ implies $x_1(a) = x_{a1}$, and vice versa. Owing to the property (14), the subspace N is complementary to $S(t)$ for all t . Here are the projector matrices $P_S(t)$ onto $S(t)$ along N and P onto N^\perp along N :

$$P_S = \begin{bmatrix} I_r & 0 \\ -F_{22}^{-1}F_{21} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

All IVPs

$$Ex' + Fx = q, \quad x(a) - x_a \in N,$$

are uniquely solvable, and each solution satisfies $x_1(a) = x_{a1}$, the subspace N plays the same role as N_{can} in Subsection 2.1, so that $N_{can} = N$, $\Pi_{can} = P_S$.

We underline the uniqueness of N_{can} , which means that the condition $x(a) - x_a \in \tilde{N}$ with any $(m - r)$ -dimensional subspace \tilde{N} other than N involves terms $x_2(a) - x_{a2}$, thus parts of $q(a)$. In particular, the condition $x(a) - x_a \in S(a)^\perp$, or equivalently $C(a)^*(x(a) - x_a) = 0$, yields⁶

$$C(a)^*(x(a) - x_a) = x_1(a) - x_{a1} - (F_{22}(a)^{-1}F_{21}(a))^*(x_2(a) - x_{a2}) = 0.$$

This will be an accurate initial condition, only if $F_{21} = 0$, but then $S(a)^\perp$ coincides with N .

It is evident that accurate initial conditions should precisely determine the value $x_1(a)$, what can be done by one of the conditions

$$x(a) - x_a \in N = N_{can}(a), \tag{16}$$

$$P(x(a) - x_a) = 0, \tag{17}$$

$$G_a(x(a) - x_a) = 0, \tag{18}$$

with a matrix G_a such that $\ker G_a = N_{can}$, $\text{rank } G_a = d$. For arbitrary given $x_a \in \mathbb{R}^m$, each of these condition yields $x_1(a) = x_{a1}$. And this underlines the role of the canonical subspace $N = N_{can}$.

Remark 2.1. Like [25], most work on DAEs assumes continuously differentiable solutions. We will do the same in the present paper in order to include all relevant approaches, in particular the one of [25] which is especially easy to understand. This means that we must assume somewhat higher smoothness of the data than in those concepts like [18] where sharp solvability statements with the lowest possible smoothness properties are required. The difference is already visible in the solution representation (15): To have x_2 not only continuous but continuously differentiable, $F_{22}^{-1}F_{21}$ and $F_{22}^{-1}q$ must be continuously differentiable. In concepts with properly involved derivative, one also accepts solution components x_2 being only continuous, thus merely continuous data.

We point out that the matrix $C(t)$ serves as a basis of the subspace $S(t)$. The assumption of a continuously differentiable solution goes along with the assumption that S is not only a continuous but also a continuously differentiable subspace varying in \mathbb{R}^m . Such smooth bases form the starting point on each level of the reduction procedure in [25].

Taking into account the possible lower smoothnesses in the reduction steps, for example as in [16], requires enormous technical effort that is hiding the essential principles.

3. Regular time-varying DAEs and their canonical subspaces

We turn to the ordered pair $\{E, F\}$ of matrix functions $E, F : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ being sufficiently smooth, at least continuous, and consider the associated DAE

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{19}$$

as well as the accompanying time-varying subspaces in \mathbb{R}^m ,

$$N(t) = \ker E(t), \quad S(t) = \{z \in \mathbb{R}^m : F(t)z \in \text{im } E(t)\}, \quad t \in \mathcal{I}. \tag{20}$$

⁶With an orthonormal basis C for S and the ansatz $x = Cz + (I - CC^*)x$ one can obtain a regular ODE for z together with uniquely solvable IVPs [23, Remarr 5.2]. The equation declared as *essential underlying implicit ODE* in [21], for example, is based on such an approach. Then, only as far as homogeneous DAEs are concerned, the initial condition $C(a)^*(x(a) - x_a) = 0$ makes sense.

In accordance with Section 2 we denote the subspace containing the flow of the homogeneous DAE at time \bar{t} by $S_{can}(\bar{t})$, that is, the set of all possible function values $x(\bar{t})$ of solutions of the DAE $Ex' + Fx = 0$,

$$S_{can}(\bar{t}) := \{\bar{x} \in \mathbb{R}^m : \text{there is a solution } x : (\bar{t} - \delta, \bar{t} + \delta) \cap \mathcal{I} \rightarrow \mathbb{R}^m \\ \text{of the homogeneous DAE such that } x(\bar{t}) = \bar{x}\}, \quad \bar{t} \in \mathcal{I}.$$

We call the canonical time-varying subspace S_{can} the *flow-subspace* of the DAE. The second canonical subspace N_{can} is, if it exists, by definition, a particular pointwise complement to S_{can} ,

$$\mathbb{R}^m = N_{can}(\bar{t}) \oplus S_{can}(\bar{t}), \quad \bar{t} \in \mathcal{I},$$

such that each IVP,

$$Ex' + Fx = q, \quad x(\bar{t}) - \bar{x} \in N_{can}(\bar{t}), \quad (21)$$

with $\bar{t} \in \mathcal{I}$, $\bar{x} \in \mathbb{R}^m$, and sufficiently smooth q , is uniquely solvable without any consistency conditions for q or its derivatives at the point \bar{t} . This subspace will be further specified in Section 5 below. We call N_{can} the *canonical complement to the flow-subspace*.

In the present section we will agree on what *regular* DAEs are, and show that then the time-varying subspace $S_{can}(\bar{t})$ is well-defined on all \mathcal{I} , and has constant dimension.

In Section 5 we will see that in case of a regular DAE both canonical subspaces are well-defined with dimensions independent of $\bar{t} \in \mathcal{I}$. The associated projector function thus becomes a generalization of the spectral projector for regular matrix pencils in Subsection 2.1.

Definition 3.1. The pair $\{E, F\}$ and the DAE (19), respectively, are called *pre-regular* on \mathcal{I} if

$$\text{im}[E(t) F(t)] = \mathbb{R}^m, \quad \text{rank} E(t) = r, \quad \dim N(t) \cap S(t) = \theta, \quad t \in \mathcal{I},$$

with integers $0 \leq r \leq m$ and $\theta \geq 0$. Additionally, if $\theta = 0$ and $r < m$, then the DAE is called *regular with index one*, but if $\theta = 0$ and $r = m$, then the DAE is called *regular with index zero*.

We underline that any pre-regular pair $\{E, F\}$ features three subspaces $S(t)$, $N(t)$, and $N(t) \cap S(t)$ having constant dimensions r , $m - r$, and θ , respectively.

We emphasize and keep in mind that now not only the coefficients are time dependent, but also the resulting subspaces. Nevertheless, we suppress in the following mostly the argument t , for the sake of better readable formulas. The equations and relations are then meant pointwise for all arguments.

The different cases for $\theta = 0$ are well-understood. A regular index-zero DAE is actually a regular implicit ODE and $S_{can} = S = \mathbb{R}^m$, $N_{can} = N = \{0\}$. Regular index-one DAEs feature $S_{can} = S$, $N_{can} = N$, e.g., [7, 18], also Section 2.2. Note that $r = 0$ leads to $S_{can} = \{0\}$. All these cases are only interesting here as intermediate results.

We turn back to the general case and describe the flow-subspace S_{can} , and end up with a regularity notion associated with a regular flow.

The pair $\{E, F\}$ is supposed to be *pre-regular*. The first step of the reduction procedure from [25] is then well-defined, we refer to [25, Section 12] for the substantiating arguments. Here we apply this procedure to homogeneous DAEs only.

We start by $E_0 = E$, $F_0 = F$, $m_0 = m$, $r_0 = r$, $\theta_0 = \theta$, and consider the homogeneous DAE

$$E_0 x' + F_0 x = 0.$$

By means of a basis $Z_0 : \mathcal{I} \rightarrow \mathbb{R}^{m_0 \times (m_0 - r_0)}$ of $(\text{im } E_0)^\perp = \ker E_0^*$ and a basis $Y_0 : \mathcal{I} \rightarrow \mathbb{R}^{m_0 \times r_0}$ of $\text{im } E_0$ we divide the DAE into the two parts

$$Y_0^* E_0 x' + Y_0^* F_0 x = 0, \quad Z_0^* F_0 x = 0.$$

From $\text{im}[E_0, F_0] = \mathbb{R}^m$ we derive that $\text{rank } Z_0^* F = m_0 - r_0$, and hence the subspace $S_0 = \ker Z_0^* F$ has dimension r_0 . Obviously, each solution of the homogeneous DAE must stay in the subspace S_0 . Choosing a continuously differentiable basis $C_0 : \mathcal{I} \rightarrow \mathbb{R}^{m_0 \times r_0}$ of S_0 , each solution of the DAE can be represented as $x = C_0 x_{[1]}$, with a function $x_{[1]} : \mathcal{I} \rightarrow \mathbb{R}^{r_0}$ satisfying the reduced to size $m_1 = r_0$ DAE

$$Y_0^* E_0 C_0 x'_{[1]} + Y_0^* (F_0 C_0 + E_0 C'_0) x_{[1]} = 0.$$

Denote $E_1 = Y_0^* E_0 C_0$ and $F_1 = Y_0^* (F_0 C_0 + E_0 C'_0)$ which have size $m_1 \times m_1$. The pre-regularity assures that E_1 has constant rank $r_1 = r_0 - \theta_0 \leq r_0$. Namely, we have

$$\ker E_1 = \ker E_0 C_0 = C_0^+ (\ker E_0 \cap S_0), \quad \dim \ker E_1 = \dim(\ker E_0 \cap S_0) = \theta_0.$$

Here, $C_0(t)^+$ denotes the Moore-Penrose generalized inverse of $C_0(t)$.

Next we repeat the reduction step supposing that the new pair $\{E_i, F_i\}$ is pre-regular again, and so on. The pair $\{E_i, F_i\}$ has size $m_i := r_{i-1}$ and E_i has rank $r_i = r_{i-1} - \theta_{i-1}$. This yields $m \geq r_0 \geq \dots \geq r_j \geq r_{j-1} \geq \dots \geq 0$. Denote by μ the smallest integer such that either $r_{\mu-1} = r_\mu > 0$ or $r_{\mu-1} = 0$. Then, it follows that $\ker E_{\mu-1} \cap S_{\mu-1} = \{0\}$, which means in turn that

$$E_{\mu-1} x'_{[\mu-1]} + F_{\mu-1} x_{[\mu-1]} = 0$$

represents a regular index-1 DAE. If $r_{\mu-1} = 0$, that is $E_{\mu-1} = 0$, then $F_{\mu-1}$ is nonsingular, which leads to a zero flow $x_{[\mu-1]}(t) \equiv 0$. On the other hand, if $r_\mu > 0$ then E_μ remains nonsingular and

$$E_\mu x'_{[\mu]} + F_\mu x_{[\mu]} = 0$$

is an implicit regular ODE living in \mathbb{R}^{m_μ} , $m_\mu = r_{\mu-1}$. Finally, if $r_\mu > 0$ each solutions of the original homogeneous DAE (19) has the form

$$x = \underbrace{C_0 C_1 \dots C_{\mu-1}}_{=: C} x_{[\mu]} =: C x_{[\mu]}, \quad C : [a, b] \rightarrow \mathbb{R}^{m_\mu \times r_{\mu-1}}, \quad \text{rank } C = r_{\mu-1}.$$

Otherwise, if $r_\mu = 0$, there is only the identically vanishing solution of the homogeneous DAE, $x = 0$.

Moreover, for each $\bar{t} \in \mathcal{I}$ and each $z \in \text{im } C(\bar{t})$, there is exactly one solution of the original homogeneous DAE passing through, $x(\bar{t}) = z$.

As proved in [25], the ranks $r = r_0 > r_1 > \dots > r_{\mu-1}$ are independent of the special choice of the involved basis functions.

The property of pre-regularity does not necessarily carry over to the subsequent pair, as Example 3.2 shows.

Example 3.2. The pair $\{E, F\}$ from [7, p. 91],

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad F(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

is pre-regular with $m = 2$, $r = 1$ and $\theta = 1$. Choosing bases

$$C_0 = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad Z_0 = \begin{bmatrix} 1 \\ -t \end{bmatrix},$$

the subsequent pair is given by $m_1 = r = 1$, $E_1 = Y_0^* E C_0 = 0$, and $F_1 = Y_0^* F C_0 + Y_0^* E C_0' = 0$, such that $[E_1 F_1]$ fails to have full row-rank m_1 , thus the pair $\{E_1, F_1\}$ fails to be pre-regular. The associated to $\{E, F\}$ homogeneous DAE possess the solutions

$$x(t) = \alpha(t) \begin{bmatrix} t \\ 1 \end{bmatrix},$$

in which α stands for an arbitrary smooth real function, which does not fit our idea of regularity. \square

Definition 3.3. The pre-regular pair $\{E, F\}$ with $r < m$ and the associated DAE (19), respectively, are called *regular*⁷ if there is an integer $\mu \in \mathbb{N}$ such that the above reduction procedure is well-defined up to level $\mu - 1$, each pair $\{E_i, F_i\}$, $i = 0, \dots, \mu - 1$, is pre-regular, and if $r_{\mu-1} > 0$ then E_μ is well-defined and nonsingular, $r_\mu = r_{\mu-1}$. If $r_{\mu-1} = 0$ we set $r_\mu = r_{\mu-1}$. The index μ and the ranks $r = r_0 > r_1 > \dots > r_{\mu-1} = r_\mu$ are called *characteristic values* of the pair and the DAE, respectively.

By construction, for a regular pair it follows that $r_{i+1} = r_i - \theta_i$, $i = 0, \dots, \mu - 1$. Therefore, in place of the above $\mu + 1$ rank values r_0, \dots, r_μ , the following rank and dimensions,

$$r \quad \text{and} \quad \theta_0 \geq \theta_1 \geq \dots \geq \theta_{\mu-2} > \theta_{\mu-1} = 0, \quad (22)$$

can serve as characteristic quantities. Later it will become clear that these data also play a supporting role in other concepts, too.

Remark 3.4. A predecessor version of the reduction procedure in [25] was already proposed and analyzed in [5] under the name *elimination of the unknowns*, even for more general pairs of rectangular matrix functions. There, an additional scaling of the respective pairs is incorporated to put them in partitioned form on each stage, cf. Appendix A.3, but this makes the description less clear. The regularity notion given in [5] is consistent with Definition 3.3. Another very related such reduction technique has been presented and extended a few years ago under the name *dissection concept* [16], see also Appendix A.3. This notion of regularity also agrees with Definition 3.3. As we shall see below, the regularity notions related to the strangeness-index concept and the tractability-index framework are consistent with Definition 3.3, too. Furthermore, the understanding of regular points e.g. in [26, Section 2.2.7] fits to this then also.

Theorem 3.5. *If the DAE (19) is regular on \mathcal{I} with index μ and characteristics*

$$r = r_0 > r_1 > \dots > r_{\mu-1},$$

or, equivalently, (22), then $S_{can}(t) \subset \mathbb{R}^m$ has dimension $d = r - \sum_{j=0}^{\mu-2} \theta_j = r_{\mu-1}$ for all $t \in \mathcal{I}$, and the matrix function $C : \mathcal{I} \rightarrow \mathbb{R}^{m \times d}$, $C = C_0 \dots C_{\mu-2}$, generated by the reduction procedure is a basis of S_{can} .

Proof. Regarding the relation $r_{i+1} = r_i - \theta_i$, $i = 0, \dots, \mu - 2$ directly resulting from the reduction procedure, the assertion is an immediate consequence of [25, Theorem 13.3]. \square

Remark 3.6. The subspaces $S_j = \ker Z_j^* F_j = \text{im } C_j \subset \mathbb{R}^{m_j}$, with $m_j = r_{j-1} = \text{rank } E_{j-1}$, $j = 1, \dots, \mu - 1$, are living in spaces of different dimension m_j . In contrast, letting

$$\begin{aligned} S^{[0]} &:= S_0, \\ S^{[j]} &:= \text{im } C_0 C_1 \dots C_j = C_0 C_1 \dots C_{j-1} \text{im } C_j = C_0 C_1 \dots C_{j-1} S_j, \quad j = 1, \dots, \mu - 1, \end{aligned}$$

⁷In [25] instead the term *completely regular* is used whereas pre-regular pairs like example 3.2 are called *regular*. We have not taken up this notation, but that of other works, which seems more suitable to us.

we arrive at the sequence of subspaces living all in \mathbb{R}^m ,

$$S_0 = S^{[0]} \supset S^{[1]} \supset \dots \supset S^{[\mu-1]},$$

showing dimensions $r = r_0 > r_1 > \dots > r_{\mu-1}$, respectively.⁸ Regarding that

$$\ker E_0 = \ker E, \quad \ker E_j = \ker EC_0 \cdots C_{j-1} = (C_0 \cdots C_{j-1})^+ \ker E, \quad j = 1, \dots, \mu - 1,$$

where $(\cdot)^+$ denotes the pointwise Moore-Penrose generalized inverse. we derive the intersections $\ker E_j \cap S_j = (C_0 \cdots C_{j-1})^+ \ker E \cap S^{[j]}$, and also the sequence of inclusions

$$S_0 \cap \ker E = S^{[0]} \cap \ker E \supset S^{[1]} \cap \ker E \supset \dots \supset S^{[\mu-1]} \cap \ker E, \quad (23)$$

with

$$\begin{aligned} \dim S^{[j]} \cap \ker E &= \dim S_j \cap \ker E_j = \dim \ker E_{j+1} = r_j - r_{j+1} = \theta_j, \\ j &= 0, \dots, \mu - 1 \end{aligned}$$

These inclusions seem to indicate a certain relationship to the approach in [6]. □

It is important to mention that pre-regularity and regularity persist and the characteristic values are invariant under equivalence transformations. An equivalence transformation of $\{E, F\}$ is given by matrix functions $L : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ being continuous and pointwise nonsingular and $K : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ being continuously differentiable and pointwise nonsingular yielding the pair \tilde{E}, \tilde{F} ,

$$\tilde{E} = LEK, \quad \tilde{F} = LFK + LEK'. \quad (24)$$

Obviously, if $\{E, F\}$ is pre-regular, then it follows $\tilde{N} = K^{-1}N$, $\tilde{S} = K^{-1}S$, and $\tilde{N} \cap \tilde{S} = K^{-1}(N \cap S)$, thus $\tilde{r} = r$, $\tilde{\theta} = \theta$. For the proof concerning regularity we refer to [25].

4. On further regularity notions and their relations

We are concerned here with the regularity notions and approaches from [5, 16, 17, 18] associated with the elimination procedure, the dissection concept, the strangeness reduction, and the tractability framework compared to Definition 3.3. The approaches in [5, 16, 17] are de facto special solution methods including reduction steps by elimination of variables and differentiations of certain variables. In contrast, the concept in [18] aims at a structural projector-based decomposition of the given DAE in order to analyse them subsequently.

We have already mentioned in Remark 3.4 that the elimination procedure in [5] is an earlier, but less elegant version of the reduction procedure in [25], which we have adopted in Section 3. The regularity definition [5, p. 58] agrees with Definition 3.3 in the matter and also with the name. However, it does not yet specify any characteristic values, which is why we will only refer to [25] in the remainder of this paper.

Each of the concepts is associated with a sequence of pairs of matrix functions, each supported by certain rank conditions that look very different. Thus also the regularity notions, which require in each case that the sequences are well-defined with well-defined termination, look very different. At the end of this section, we will know that all these regularity terms agree with our Definition 3.3, and that the characteristics (22) capture all the rank conditions involved.

When describing the different methods, traditionally the same terms are used, for example $\{E_j, F_j\}$ for the matrix function pairs and r_j for the ranks. However, they have different meanings in each instance.

⁸Note that this sequence is closely related to the first Wong chain in [1].

To avoid confusion, we label the different characters with corresponding exponents R (reduction), S (strangeness), D (dissection), and T (tractability), respectively, when there is a risk of confusion.

We first relate the regularity notion given by Definition 3.3 to the strangeness reduction concept.

Within the strangeness reduction framework the following five rank-values of the matrix function pair $\{E, F\}$ play their role, e.g., [17, p. 59]:

$$r = \text{rank } E, \quad (25)$$

$$a = \text{rank } Z^*FT, \quad (26)$$

$$s = \text{rank } V^*Z^*FT^c, \quad (27)$$

$$d = r - s, \quad (28)$$

$$u = m - r - a - s, \quad (29)$$

whereby T, T^c, Z, V represent orthonormal bases (ONBs) of $\ker E$, $(\ker E)^\perp$, $(\text{im } E)^\perp$, and $(\text{im } Z^*FT)^\perp$, respectively. The strangeness concept is tied to the requirement that r, a , and s are well-defined constant integers.

Lemma 4.1. *The pair $\{E, F\}$ is pre-regular, if and only if the values (25)-(29) are constant and $u = 0$. In case of pre-regularity, one has*

$$a = m - r - \theta, \quad s = \theta, \quad d = r - \theta.$$

Proof. Let $\{E, F\}$ be pre-regular, and T, T^c be smooth ONBs what is possible owing to the constant rank r of E . Let Z, V be pointwise ONBs and N, S be given by (20). Then it holds $S = \ker Z^*F$ by construction, and Z^*FT has size $(m - r) \times (m - r)$. From $\ker Z^*FT = T^*(N \cap S)$ we derive $\dim \ker Z^*FT = \theta$ and hence

$$a = \text{rank } Z^*FT = m - r - \theta.$$

Next we investigate the strangeness value s . For this aim we decompose

$$S = (N \cap S) \oplus X,$$

with $X \subset S$, $\dim X = r - \theta$ and choose a basis C_X of X so that $\text{im } C_X = X$, $\text{rank } C_X = m - \theta$, $\text{im } C_X \cap N = \{0\}$.

Since $T^{c*}C_X y = 0$ means $C_X y \in \ker E = N$, thus $C_X y = 0$, and therefore $y = 0$, we learn that

$$\text{rank } T^{c*}C_X = r - \theta.$$

Regarding the decomposition

$$\mathbb{R}^{m-r} = \text{im } Z^*FT \oplus (\text{im } Z^*FT)^\perp = \text{im } Z^*FT \oplus \text{im } V$$

we conclude $\text{rank } V = \theta$.

Now we inspect the nullspace of $V^*Z^*FT^c$. If $z \in \mathbb{R}^r$ and $V^*Z^*FT^c z = 0$, thus $Z^*FT^c z \in \text{im } Z^*FT$, then there is a $w \in \mathbb{R}^{m-r}$ such that $Z^*FT^c z = -Z^*FT w$, further

$$Z^*F[T^c T] \begin{bmatrix} z \\ w \end{bmatrix} = 0, \quad \text{that is,} \quad [T^c T] \begin{bmatrix} z \\ w \end{bmatrix} \in S = (N \cap S) \oplus \text{im } C_X.$$

This yields $T^c z + T w = \xi + C_X v$ with a $\xi \in N \cap S$ and a $v \in \mathbb{R}^{r-\theta}$. Taking into account that T and T^c are ONBs of N and N^\perp , respectively, we conclude $z = T^{c*}T^c z = T^{c*}C_X v$, and hence

$$\text{im } T^{c*}C_X \subseteq \ker V^*Z^*FT^c.$$

On the other hand, choosing $z \in \text{im } T^{c*}C_X$ we compute

$$\begin{aligned} V^*Z^*FT^c z &= V^*Z^*FT^c T^{c*}C_X v \\ &= V^*Z^*F(I - TT^*)C_X v \\ &= V^*Z^*FC_X v - V^*Z^*FTT^*C_X v = 0, \end{aligned}$$

yielding

$$\text{im } T^{c*}C_X = \ker V^*Z^*FT^c, \quad \text{rank } V^*Z^*FT^c = r - \text{rank } T^{c*}C_X = r - (r - \theta) = \theta.$$

Finally, we obtain $a + s = m - r - \theta + \theta = m - r$ and $u = 0$.

On the other hand, let the pair $\{E, F\}$ have constant rank values (25)–(29), and $u = 0$. Applying the basic arguments of the strangeness reduction [17, p. 68f] we transform the pair $\{E, F\}$ equivalently to $\{\mathring{E}, \mathring{F}\}$,

$$\mathring{E} = \begin{bmatrix} I_s & & & \\ & I_d & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad \mathring{F} = \begin{bmatrix} 0 & \tilde{F}_{12} & 0 & \tilde{F}_{14} \\ 0 & 0 & 0 & \tilde{F}_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \end{bmatrix},$$

with $d + s = r$, $a + s = m - r$.

For better clarity of the following we add a further permutation transformation. Using the permutation matrix

$$K_P = \begin{bmatrix} 0 & 0 & 0 & I_s \\ 0 & I_d & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \end{bmatrix},$$

we arrive at

$$\tilde{E} = \mathring{E}K_P = \begin{bmatrix} 0 & 0 & 0 & I_s \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{F} = \mathring{F}K_P = \begin{bmatrix} \tilde{F}_{14} & \tilde{F}_{12} & 0 & 0 \\ \tilde{F}_{24} & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix}, \quad (30)$$

which is, of course, again equivalent to the original pair $\{E, F\}$. It remains checking its pre-regularity. The condition $\text{im}[\tilde{E} \tilde{F}] = \mathbb{R}^m$ is evident and \tilde{E} has constant rank $r = s + d$. From

$$\tilde{N} = \ker \tilde{E} = \text{im} \begin{bmatrix} I_s & 0 \\ 0 & 0 \\ 0 & I_a \\ 0 & 0 \end{bmatrix}, \quad \tilde{S} = \text{im} \tilde{C}, \quad \tilde{C} = \begin{bmatrix} I_s & 0 \\ 0 & I_d \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{N} \cap \tilde{S} = \text{im} \begin{bmatrix} I_s \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (31)$$

we know that $\theta = s$, such that the pair $\{\tilde{E}, \tilde{F}\}$, and in turn the original pair $\{E, F\}$, is pre-regular. \square

Assuming that the pair $\{E, F\}$ is pre-regular, in turn the transformed pair $\{\tilde{E}, \tilde{F}\}$ given by (30) is also pre-regular, we provide the successor pairs $\{\tilde{E}_1^R, \tilde{F}_1^R\}$ according to the reduction procedure in Section 3 and $\{E_1^S, F_1^S\}$ according to the strangeness framework [17].

Using the basis functions \tilde{Y} and \tilde{C} ,

$$\tilde{Y}^* = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I_s & 0 \\ 0 & I_d \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

we form the reduced pair $\{\tilde{E}_1^R, \tilde{F}_1^R\}$ to $\{\tilde{E}, \tilde{F}\}$ in accordance with Section 3, that is,

$$\tilde{E}_1^R = \tilde{Y}^* \tilde{E} \tilde{C} = \begin{bmatrix} 0 & 0 \\ 0 & I_d \end{bmatrix}, \quad \tilde{F}_1^R = \tilde{Y}^* \tilde{F} \tilde{C} = \begin{bmatrix} \tilde{F}_{14} & \tilde{F}_{12} \\ \tilde{F}_{24} & 0 \end{bmatrix}.$$

The matrix function \tilde{E}_1^R has size $r \times r$, $r = s + d$ and constant rank $r_{[1]} = d$. If we look at the subspaces,

$$\tilde{S}_1^R = \{z \in \mathbb{R}^{s+d} : \tilde{F}_{14}z_1 + \tilde{F}_{12}z_2 = 0\}, \quad \tilde{S}_1^R \cap \tilde{E}_1^R = \{z \in \mathbb{R}^{s+d} : \tilde{F}_{14}z_1 = 0, z_2 = 0\},$$

it becomes clear that the reduced pair $\{\tilde{E}_1^R, \tilde{F}_1^R\}$ is pre-regular again, if and only if the following two conditions,

$$\text{im}[\tilde{F}_{14} \tilde{F}_{12}] = \mathbb{R}^s, \quad (32)$$

$$\theta_1 := \dim \tilde{S}_1^R \cap \ker \tilde{E}_1^R = \dim \ker \tilde{F}_{14} \quad \text{is constant.} \quad (33)$$

are satisfied. In particular, (33) requires that the $s \times s$ matrix function \tilde{F}_{14} shows constant rank $s - \theta_1 = \theta - \theta_1$.

On the other hand, the matrix function pair $\{E_1^S, F_1^S\}$ following the original pair $\{E, F\}$ within the strangeness framework is given by replacing the entry $(\tilde{E})_{14}$ in (30) by a zero matrix, which leads to

$$E_1^S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_1^S = \tilde{F} = \begin{bmatrix} \tilde{F}_{14} & \tilde{F}_{12} & 0 & 0 \\ \tilde{F}_{24} & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix}.$$

We determine the strangeness characteristics of this pair. For this aim we use the corresponding bases

$$T_1 = \begin{bmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_1^c = \begin{bmatrix} 0 \\ I_d \\ 0 \\ 0 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_s \end{bmatrix}.$$

and obtain

$$Z_1^* F_1^S T_1 = \begin{bmatrix} \tilde{F}_{14} & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_s \end{bmatrix}, \quad Z_1^* F_1^S T_1^c = \begin{bmatrix} \tilde{F}_{12} \\ 0 \\ 0 \end{bmatrix},$$

and further $r_1^S = d = r - \theta = r_{[1]}$ and $a_1 = \text{rank } Z_1^* F_1^S T_1 = a + s + \text{rank } \tilde{F}_{14}$. Set $r_{F14} = \text{rank } F_{14}$. Let \tilde{V}_{14} denote an ONB of $(\text{im } \tilde{F}_{14})^\perp$. \tilde{V}_{14} has size $s \times (s - r_{F14})$. Then, the matrix function

$$V_1 = \begin{bmatrix} \tilde{V}_{14} \\ 0 \\ 0 \end{bmatrix}, \quad \text{with size } (s + a + s) \times (s - r_{F14})$$

forms an ONB of $(\text{im } Z_1^* F_1^S T_1)^\perp$. Next we compute

$$V_1^* Z_1^* F_1^S T_1^c = \tilde{V}_{14}^* \tilde{F}_{12}, \quad s_1 = \text{rank } \tilde{V}_{14}^* \tilde{F}_{12}.$$

further $u_1 = m - r_1^S - a_1 - s_1 = m - r + s - (a + s + r_{F14}) - s_1 = s - r_{F14} - s_1$. This results in the fact that the pair $\{E_1^S, F_1^S\}$ is pre-regular, exactly if r_{F14} is constant, in turn a_1 is constant, and further $u_1 = 0$, so that $s_1 = s - r_{F14} = \dim \ker \tilde{F}_{14}$. This requires exactly the conditions (32) and (33). In the consequence, the pairs $\{\tilde{E}_1^R, \tilde{F}_1^R\}$ and $\{E_1^S, F_1^S\}$ are pre-regular simultaneously. If they are pre-regular, then $s_1 = \theta_1$, $a_1 = m - r + s - s_1 = m - r + \theta - \theta_1$.

Lemma 4.2. *Given are two matrix function pairs $\{E, F\}$ and $\{\bar{E}, \bar{F}\}$ acting in \mathbb{R}^m and $\mathbb{R}^{\bar{m}}$, respectively, and $\bar{m} < m$,*

$$E = \begin{bmatrix} \bar{E} & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \bar{F} & 0 \\ 0 & I_{m-\bar{m}} \end{bmatrix}.$$

Then $\{E, F\}$ is pre-regular if and only if $\{\bar{E}, \bar{F}\}$ is so. If they are pre-regular, the characteristics are

$$r = \bar{r}, a = \bar{a} + (m - \bar{m}), s = \bar{s}, \theta = \bar{\theta}, s = \theta.$$

Proof. Obviously, E and \bar{E} share their rank, and $\text{im}[\bar{E} \ \bar{F}] = \mathbb{R}^{\bar{m}}$ implies $\text{im}[E \ F] = \mathbb{R}^m$ and vice versa. From

$$\begin{aligned} \bar{S} &= \{w \in \mathbb{R}^{\bar{m}} : \bar{F}w \in \text{im} \bar{E}\}, \\ S &= \{z \in \mathbb{R}^{\bar{m}+(m-\bar{m})} : \bar{F}z_1 \in \text{im} \bar{E}, z_2 = 0\} = \bar{S} \times \{0\}, \\ N &= \bar{N} \times \mathbb{R}^{m-\bar{m}}, \\ N \cap S &= (\bar{N} \cap \bar{S}) \times \{0\}, \end{aligned}$$

we conclude that the pairs are pre-regular simultaneously. The 0's in the letter identities belong to $\mathbb{R}^{m-\bar{m}}$. If the matrix pairs are pre-regular, then obviously, $r = \bar{r}, \theta = \bar{\theta}$.

If $\bar{T}, \bar{T}^c, \bar{Z}$ are ONBs to $\ker \bar{E}, (\ker \bar{E})^\perp, (\text{im} \bar{E})^\perp$ then

$$T = \begin{bmatrix} \bar{T} & 0 \\ 0 & I_{m-\bar{m}} \end{bmatrix}, \quad T^c = \begin{bmatrix} \bar{T}^c \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} \bar{Z} & 0 \\ 0 & I_{m-\bar{m}} \end{bmatrix},$$

are ONBs to $\ker E, (\ker E)^\perp, (\text{im} E)^\perp$. This leads to

$$Z^*FT = \begin{bmatrix} \bar{Z}^*\bar{F}\bar{T} & 0 \\ 0 & I_{m-\bar{m}} \end{bmatrix}, \quad a = \text{rank} Z^*FT = \text{rank} \bar{Z}^*\bar{F}\bar{T} + m - \bar{m} = \bar{a} + m - \bar{m}.$$

Finally, owing to Lemma 4.1 we find $s = \theta, \bar{s} = \bar{\theta}$, and hence $s = \bar{s}$. □

Theorem 4.3. *Let the pair $\{E, F\}$ be regular on \mathcal{S} with index $\mu \in \mathbb{N}$ and characteristics $r < m, \theta_0 = 0$ if $\mu = 1$, and, for $\mu > 1$,*

$$r < m, \quad \theta_0 \geq \dots \geq \theta_{\mu-2} > \theta_{\mu-1} = 0.$$

Then the following statements are true:

(a) *The strangeness index μ_S is well-defined for $\{E, F\}$, and $\mu_S = \mu - 1$. The associated characteristics are⁹*

$$\begin{aligned} r_0^S &= r, \\ s_i &= \theta_i, \\ d_i &= r_i^S - \theta_i, \\ a_i &= m - r_i - \theta_i, \\ u_i &= 0 \\ r_{i+1}^S &= d_i, \quad i = 0, \dots, \mu - 1. \end{aligned}$$

(b) *The pair $\{E, F\}$ is regular with tractability index μ and characteristics*

$$r_0^T = r, \quad r_i^T = m - \theta_{i-1}, \quad i = 1, \dots, \mu.$$

⁹The quantities a_i, d_i, s_i, u_i only occur in the strangeness concept and do not require the S label.

(c) The pair $\{E, F\}$ is regular with dissection index μ and characteristics

$$r_0^D = r, \quad r_i^D = m - \theta_{i-1}, \quad i = 1, \dots, \mu.$$

Proof. We emphasize again that all characteristics involved here do not change under equivalence transformations [25, 17, 18, 16].

(a): We perform the reduction process of the strangeness concept step by step and compare each level with the reduction from [25]. With the help of Lemmata 4.1 and 4.2 we get step by step the assertion. The reduction pairs in [25], which have lower dimension, turn out to be (to equivalence exactly) those parts of the pairs from [17] which still play a role for the further.

(b): Given Assertion (a), Assertion (b) is a direct consequence of [18, Theorem 2.79]. We refer to Appendix A.4 for informations concerning the admissible matrix function sequences associated with the tractability index.

(c): Given Assertion (b), Assertion (c) is a direct consequence of [16, Theorem 4.25]. We refer to Appendix A.3 for a brief description of the basic step in the dissection index concept. \square

Corollary 4.4. *Let the pair $\{E, F\}$ of matrices $E, F \in \mathbb{R}^{m \times m}$ be regular in the sense of Definition 3.3 with characteristic values (22). Then the matrix pencil $\lambda E + F$ is regular and, for $i = 0, \dots, \mu - 2$, the quantity θ_i is the number of Jordan blocks of order $\geq 2 + i$ within the nilpotent matrix \mathcal{N} in the Weierstraß–Kronecker form (3).*

Proof. A constant matrix pair and the associated matrix pencil are known to be regular with Kronecker index μ exactly if it is regular with tractability index μ , e.g., [18, Chapter 1], and if so, $l_i = m - r_{i-1}^T$ is the number of Jordan blocks of order $\geq i$ in the nilpotent matrix \mathcal{N} in (3), $i = 0, \dots, \mu - 2$. Owing to Theorem 4.3 we find that $l_i = m - r_{i-1}^T = m - (m - \theta_{i-2}) = \theta_{i-2}$, for $i = 2, \dots, \mu$. \square

Remark 4.5. It seems to us very worth highlighting, that the θ -characteristics (22) make sense in all approaches considered here and allow to determine all further characteristic values. Additionally, Corollary 4.4 reveals a feature independent of any method. Furthermore, so far it is clear that θ_i coincides with s_i and it is the dimension of intersecting subspaces in different approaches¹⁰:

$$\begin{aligned} \ker E_i^R \cap S_i^R &= \ker E_i^R \cap \{z \in \mathbb{R}^{m_i} : F_i^R z \in \text{im } E_i^R\}, \\ \ker E \cap S^{[i]}, &\quad \text{in Remark 3.6,} \\ \ker G_i \cap S_i^T &= \ker G_i \cap \{z \in \mathbb{R}^m : B_i z \in \text{im } G_i\} = \ker G_i \cap \{z \in \mathbb{R}^m : B_0 z \in \text{im } G_i\}. \end{aligned}$$

Remark 4.6. We have compared here the application of the reduction procedures from [17, 16] and [25] to homogeneous DAEs, only. If, on the other hand, inhomogeneities q shall be considered, differentiations of q must be made at each level. In [25], the resulting explicit relations are considered as being finished and, thus, neglected. So only the system of lower dimension, which is still of interest, is treated further. In contrast, in [17], all equations are further carried along. Apart from equivalence transformations at each level, this makes all the difference.

¹⁰See Appendix A.4 for the definition of G_i, B_i .

5. Accurate initial condition and the second canonical subspace N_{can}

We emphasize again that we are not looking for consistent initial values here, but for an adequate formulation of initial conditions that lead to uniquely solvable IVPs, i.e., that precisely determine the free parameters of the flow of the DAE. We adopt the notion of accurately stated boundary condition [19, Definition 2.3] for this purpose. Consider the IVP

$$Ex' + Fx = q, \quad G_a x(a) = \gamma, \quad (34)$$

for a regular DAE featuring dynamical degree of freedom d , with a matrix $G_a \in \mathbb{R}^{l \times m}$, $l \geq d$, $a \in \mathcal{I}$, $\gamma \in \text{im } G_a \subseteq \mathbb{R}^l$.

Let x_* be a solution of the IVP (34). Following [19, Definition 2.3], the initial condition in (21) is *accurately stated*, if all slightly perturbed IVPs

$$Ex' + Fx = q, \quad G_a x(a) = \gamma + \Delta\gamma, \quad \Delta\gamma \in \text{im } G_a, \quad (35)$$

are uniquely solvable and their solutions satisfy, on a compact interval $\mathcal{I}_a \subseteq \mathcal{I}$, the inequality

$$\max_{t \in \mathcal{I}_a} |x(t) - x_*(t)| \leq K |\Delta\gamma|. \quad (36)$$

It should be remembered that regular higher-index DAEs lead to ill-posed problems, even if the initial conditions are stated accurately, e.g., [24].

Regular index-1 DAEs are studied in detail in [7, 18], cf. also Subsection 2.2. Their dynamical degree of freedom is $d = r = \text{rank } E(t)$, and one has simply

$$N_{can} = N, \quad S_{can} = S$$

with N and S from (20). The related canonical projector function $\Pi_{can} : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ is given and also the solvability statements for IVPs with the initial condition

$$x(a) - x_a \in N(a) = N_{can}(a), \quad x_a \in \mathbb{R}^m,$$

are proved. We underline that here x_a is arbitrary and it is not necessarily a consistent value. On the other hand, $\Pi_{can}(a)x(a) = \Pi_{can}(a)x_a$ is always valid for the solution, while $x(a) = x_a$ cannot be expected in general.

While in the index-1 case one has simply $N_{can} = N$ and $S_{can} = S$, in the case of regular higher-index pairs the canonical subspaces will be subspaces with higher and lower dimension, respectively, and

$$N_{can} \supset N, \quad S_{can} \subset S.$$

Theorem 5.1. *Let the pair $\{E, F\}$ be regular on \mathcal{I} with index $\mu \in \mathbb{N}$ and characteristics $r < m$, $\theta_0 = 0$ if $\mu = 1$, and, for $\mu > 1$,*

$$r < m, \quad \theta_0 \geq \dots \geq \theta_{\mu-2} > \theta_{\mu-1} = 0.$$

Let S_{can} be the canonical flow subspace provided by Theorem 3.5. Then the following statements are true:

- (a) *The canonical complement¹¹ N_{can} to the flow subspace S_{can} is well-defined on the given interval \mathcal{I} . It has everywhere on \mathcal{I} the dimension $m - d = m - r + \sum_{j=0}^{\mu-2} \theta_j$, and the decomposition*

$$\mathbb{R}^m = S_{can} \oplus N_{can}$$

holds pointwise on \mathcal{I} . Additionally, the canonical projector Π_{can} onto S_{can} along N_{can} is well-defined on all \mathcal{I} .

¹¹which satisfies (21) by definition

(b) If $G_a \in \mathbb{R}^{l \times m}$ is such that $l \geq d$, $\ker G_a = N_{can}$, and if q is sufficiently smooth, then each IVP

$$Ex' + Fx = q, \quad G_a(x(a) - x_a) = 0, \quad x_a \in \mathbb{R}^m, \quad a \in \mathcal{I},$$

has a unique solution which features the relation $\Pi_{can}(a)x(a) = \Pi_{can}(a)x_a$ and satisfies, on compact intervals $\mathcal{I}_a \subseteq \mathcal{I}$ around a , the inequality

$$\max_{t \in \mathcal{I}_a} |x(t)| \leq K \{ |\Pi_{can}(a)x_a| + \max_{t \in \mathcal{I}_a} |q(t)| + \sum_{l=1}^{\mu-1} \max_{t \in \mathcal{I}_a} |q^{(l)}(t)| \}, \quad (37)$$

with a constant K depending on the pair $\{E, F\}$ and the interval \mathcal{I}_a only¹².

The assertions justify the notation N_{can} .

Proof. Owing to Theorem 4.3 the pair $\{E, F\}$ is regular with tractability index μ and characteristics

$$r_0^T = r, \quad r_i^T = m - \theta_{i-1}, \quad i = 1, \dots, \mu.$$

This means in the projector-based framework that there is an admissible matrix function sequence (see [18], also Appendix A.4 below). The related nullspaces N_i have dimensions θ_{i-1} , $i = 1, \dots, \mu$, and $\dim N_0 = m - r$, and

$$\ker \Pi_{\mu-1} = N_0 + N_1 + \dots + N_{\mu-1} = N_0 \oplus N_1 \oplus \dots \oplus N_{\mu-1} =: N_{can} \quad (38)$$

has dimension $\dim N_{can} = \sum_{i=0}^{\mu-1} \dim N_i = m - r + \sum_{i=1}^{\mu-1} \dim \theta_{i-1}$. The subspace N_{can} is shown to be independent of the special choice of the admissible projector functions which form the projector function $\Pi_{\mu-1}$, [18, Theorem 2.8]. Since we suppose sufficiently smooth E and F , a so-called fine decoupling sequence can be constructed starting with an arbitrary projector function Π_0 onto $N_0 = \ker E$. Then, for $\Pi_{\mu-1}$ associated with a fine decoupling sequence, there is a further special projector function Q_{*0} onto N_0 , such that (cf. [18, Lemma 2.31, Theorem 2.42]) $\Pi_{can} := (I - Q_{*0})\Pi_{\mu-1}$ is again a projector function. Π_{can} has constant rank d on all \mathcal{I} , and by construction $\ker \Pi_{can} = N_{can}$. It is further verified in [18, Section 2.6] that $\text{im} \Pi_{can} = S_{can}$, which in turn shows that N_{can} is actually a complementary subspace to S_{can} .

Assertion (b) is now completely verified by [18, Theorem 2.63]. □

Corollary 5.2. *Under the conditions of Theorem 5.1 concerning the pair $\{E, F\}$ and the matrix G_a , the initial condition in the IVP (34) is accurately stated.*

Proof. Since $\gamma, \Delta\gamma \in \text{im} G_a$, we may choose $x_a, \Delta x_a \in \mathbb{R}^m$ such that $\gamma = G_a x_a$, $\Delta\gamma = G_a \Delta x_a$, and the IVPs (35) are uniquely solvable by Theorem 5.1(b). Furthermore, $z := x - x_*$ satisfies the homogeneous DAE and the initial condition $G_a z(a) - G_a \Delta x_a = 0$, and (37) implies

$$\max_{t \in \mathcal{I}_a} |z(t)| \leq K |\Pi_{can}(a)\Delta x_a| = K |G_a^- G_a \Delta x_a| \leq \tilde{K} |G_a \Delta x_a| = \tilde{K} |\Delta\gamma|,$$

what was to show. □

Finally in this part, let us add that the maximal fundamental solution matrix of a regular DAE normalized at $a \in \mathcal{I}$, that is the solution of the IVP,

$$EX' + FX = 0, \quad X(a) = \Pi_{can}(a)$$

feature semigroup properties and $\text{im} X(t, a) = S_{can}(t)$, $\ker X(t, a) = N_{can}(a)$, what we could see so similarly also in the case of constant matrix pairs in Section 2.1.

¹²From (37) one immediately obtains the somewhat coarser inequality

$$|x(t)| \leq \bar{K} \{ |x(a)| + \max_{a \leq s \leq t} |q(s)| + \sum_{l=1}^{\mu-1} \max_{a \leq s \leq t} |q^{(l)}(s)| \},$$

which indicates that each regular DAE with index μ has also perturbation index μ .

6. An useful representations of the matrix G_a for accurately stated initial conditions and the projector function Π_{can}

In the framework of the projector-based analysis [18] admissible matrix function sequences and incorporated admissible projector functions play their role, see also Appendix A.4. Owing to [18, Theorem 2.8] the subspaces $N_0 + N_1 + \dots + N_i$ do not depend of the special choice of the involved projector functions. For a regular DAE with index μ , it holds that

$$\ker \Pi_{can} = \ker \Pi_{\mu-1} = N_0 \oplus N_1 \oplus \dots \oplus N_{\mu-1},$$

with $\Pi_{\mu-1}$ given by an arbitrary admissible matrix function sequence. However, the possibilities of practical calculation are still limited, so we are looking for another way.

The homogeneous adjoint DAEs to the above DAEs (1) and (2), that is, $E^*x' + Fx = q$ and $A(Dx)' + Bx = q$, are

$$-E^*y' + (F^* - E^*)y = 0, \quad (39)$$

$$-D^*(A^*y)' + B^*y = 0. \quad (40)$$

Owing to [20, Theorem 3 and Corollary 2], the original DAE and its adjoint are regular with index μ at the same time, and they share the related characteristics, in particular the dynamical degree of freedom d .

Let Π_{*can} denote the canonical projector function associated with the adjoint DAE and let $C_* : \mathcal{I} \rightarrow \mathbb{R}^{m \times d}$ be a basis of $S_{*can} = \text{im} \Pi_{*can}$, which can be provided, for example, using the procedure from Section 3.

Let's take a brief look at the semi-explicit index-1 DAE in Subsection 2.2 and consider its adjoint DAE.

Example 6.1. The pair $\{E_*, F_*\}$ below describes the DAE adjoint to the DAE in 2.2,

$$E_* = -E^* = -E = - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad F_* = F^* = \begin{bmatrix} F_{11}^* & F_{21}^* \\ F_{12}^* & F_{22}^* \end{bmatrix}, \quad \text{with } F_{22} \text{ remaining nonsingular.}$$

The pair is regular with index one, with

$$N_* = \text{im} \begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix}, \quad S_* = \text{im} C_*, \quad C_* = \begin{bmatrix} I_r \\ -F_{22}^{*-1} F_{12}^* \end{bmatrix}, \quad N_* \cap S_* = \{0\},$$

such that $\mathbb{R}^m = S_* \oplus N_*$. The canonical projector function Π_{*can} onto S_* along N_* and the orthogonal projector P_* onto N_*^\perp along N_* are given by

$$\Pi_{*can} = \begin{bmatrix} I_r & 0 \\ -F_{22}^{*-1} F_{12}^* & 0 \end{bmatrix}, \quad P_* = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Observe that evidently here $d = r$ and

$$C_*^* E = (I_d - F_{12} F_{22}^{-1}) E = [I_d \ 0],$$

$$\ker \Pi_{can} = \ker C_*^* E,$$

and hence, for stating accurate initial conditions one can choose

$$G_a(x(a) - x_a) = 0, \quad G_a = C_*^*(a)E(a)$$

However, in the index-1 case in contrast to all higher-index cases there is the simpler possibility for the given DAE, namely, by choosing $x(a) - x_a \in \ker E = N = N_{can}$. \square

Theorem 6.2. *Let the pair $\{E, F\}$ be regular with index μ and canonical subspaces S_{can} and N_{can} , with bases $C_{S_{can}}$ and $C_{N_{can}}$, respectively.*

Then the adjoint pair $\{-E^, F^* - E^{*'}\}$ is also regular with the same characteristics. Moreover, with bases $C_{S_{*can}}, C_{N_{*can}}$ of their canonical subspaces S_{*can} and N_{*can} , it results that*

$$N_{can} = \ker C_{S_{*can}}^* E, \quad N_{*can} = \ker C_{S_{can}}^* E^*.$$

Proof. We emphasize again that with the smoothness assumed here in general, it does not matter whether one chooses a homogeneous DAE in standard form, $Ex' + Fx = 0$, or a DAE with proper involved derivative, $A(Dx)' + Bx = 0$, to a given regular pair $\{E, F\}$, and $E = AD, B = F - AD'$, see Appendix A.1. The pleasant symmetry of the DAE with proper involved derivative to its adjoint in many cases facilitates the investigation, although $-D^*(A^*y)' + B^*y = 0$ is only another notation for $-E^*y' + (F^* - E^{*'})y = 0$.

By [20, Theorem 3], the adjoint pair inherits the regularity along with characteristics from the regular pair $\{E, F\}$.

Owing to [20, Lemma 3] we know that

$$D\Pi_{can}D^- = (A^*\Pi_{*can}A^{*-})^*,$$

in which D^- and A^{*-} are special generalized inverses corresponding to so-called complete decouplings. Regarding the relations

$$\ker D\Pi_{can} = \ker \Pi_{can}, \quad \text{im } \Pi_{*can}A^{*-} = \text{im } \Pi_{*can},$$

we derive

$$\ker \Pi_{can} = \ker (A^{*-})^* \Pi_{*can}^* AD = \ker \Pi_{*can}^* AD.$$

On the other hand, writing shorter $C_* = C_{S_{*can}}$ it holds that

$$\ker C_*^* = (\text{im } C_*)^\perp = (\text{im } \Pi_{*can})^\perp = \ker \Pi_{*can}^*,$$

further

$$\begin{aligned} \ker C_*^* AD &= (\text{im } (AD)^* C_*)^\perp = ((AD)^* \text{im } \Pi_{*can})^\perp = (\text{im } (AD)^* \Pi_{*can})^\perp \\ &= \ker \Pi_{*can}^* AD, \end{aligned}$$

and hence $\ker \Pi_{can} = \ker C_*^* AD$. The second relation in the assertion of the theorem is valid for symmetry arguments. \square

Having the basis $C =: C_{S_{can}}$ of S_{can} , and a basis $C_{N_{can}}$ of $\ker C_*^* AD = N_{can}$, respectively, the $m \times m$ matrix function $\mathcal{M} := [C_{S_{can}} \ C_{N_{can}}]$ remains nonsingular everywhere on \mathcal{I} , and the canonical projector is given by

$$\Pi_{can} = \mathcal{M} \begin{bmatrix} I_d & \\ & 0_{m-d} \end{bmatrix} \mathcal{M}^{-1}.$$

We conclude this section with an easy-to-follow simple example.

Example 6.3. Consider the coefficient pair

$$E(t) = \begin{bmatrix} 1 & \alpha(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 & \alpha'(t) & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad t \in \mathcal{I},$$

with an arbitrary smooth function $\alpha : \mathcal{I} \rightarrow \mathbb{R}$. Compute $r = 2$,

$$\begin{aligned} Z_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, Y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, Z_0^* F = [0 \quad 1 \quad 0], C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ker E = \text{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta_0 = 1, \\ E_1 &= Y_0^* E C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_1 = Y_0^* F C_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \\ Z_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Z_1^* F_1 = [0 \quad -1], C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ker E_1 = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \theta_0 = 0, \end{aligned}$$

which indicates that the associated DAE is regular with index $\mu = 2$. The flow subspace is simply

$$S_{can} = \text{im} C, \quad C = C_0 C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next we provide the canonical complement by means of the projector based approach. We form

$$\begin{aligned} Q_0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1 := E + F Q_0 = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \\ N_{can} &= \ker E \oplus \ker G_1 = \text{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \oplus \text{im} \begin{bmatrix} -\alpha \\ 1 \\ 1 \end{bmatrix} = \text{im} \begin{bmatrix} 0 & -\alpha \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

It is evident that $G_a = [1 \quad \alpha(a) \quad 0]$ is suitable for accurate initial condition at the point $a \in \mathcal{I}$. Next we turn to the coefficients of the adjoint DAE,

$$E_* := -E^* = - \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_* := F^* - E^{*'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad t \in \mathcal{I},$$

and apply the reduction procedure. We form

$$\begin{aligned} Z_{*,0} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, Y_{*,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, Z_{*,0}^* F_* = [0 \quad -1 \quad 0], C_{*,0} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ker E_* = \text{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta_{*,0} = 1, \\ E_{*,1} &= Y_{*,0}^* E_* C_{*,0} = \begin{bmatrix} -1 & 0 \\ -\alpha & 0 \end{bmatrix}, F_{*,1} = Y_{*,0}^* F_* C_{*,0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ Z_{*,1} &= \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}, Z_{*,1}^* F_{*,1} = [0 \quad 1], C_{*,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ker E_{*,1} = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \theta_{*,1} = 0, \end{aligned}$$

and also

$$S_{*,can} = \text{im} C_*, \quad C_* = C_{*,0} C_{*,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_*^* E = [1 \quad \alpha \quad 0].$$

Moreover, applying the projector-based approach we find

$$\begin{aligned} Q_{*,0} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{*,1} := E_* + F_* Q_{*,0} = \begin{bmatrix} -1 & 0 & 0 \\ -\alpha & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ N_{*,can} &= \ker E_* \oplus \ker G_{*,1} = \text{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \oplus \text{im} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \text{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This is consistent with the assertion of Theorem 6.2. Obviously, the canonical projector functions are simply.

$$\Pi_{can} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_{*can} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7. As an illustrating example: The linearised Campbell–Moore DAE

We investigate the linear DAE $E(t)x'(t) + F(t)x(t) = q(t)$ given by its coefficients

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F(t) = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \sin t & 0 & 1 & -\cos t & -2\rho \cos^2 t \\ 0 & 0 & -\cos t & -1 & 0 & -\sin t & -2\rho \sin t \cos t \\ 0 & 0 & 1 & 0 & 0 & 0 & 2\rho \sin t \\ 2\rho \cos^2 t & 2\rho \sin t \cos t & -2\rho \sin t & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -I_3 & 0 \\ \mathfrak{C}(t) & \mathfrak{A}(t) & \mathfrak{B}(t) \\ -\mathfrak{B}(t)^* & 0 & 0 \end{bmatrix}, \quad \rho \neq 0,$$

$$\mathfrak{A}(t) = \begin{bmatrix} 0 & 1 & -\cos t \\ -1 & 0 & -\sin t \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{B}(t) = 2\rho \begin{bmatrix} -\cos^2 t \\ -\sin t \cos t \\ \sin t \end{bmatrix}, \quad \mathfrak{C}(t) = \begin{bmatrix} 0 & 0 & \sin t \\ 0 & 0 & -\cos t \\ 0 & 0 & 1 \end{bmatrix},$$

whereby the resulting blocks of different sizes with only zero entries are all denoted by 0. It is a linearised version of a test problem proposed in [4] and has been used for numerical tests in [11, 10]. Emphasizing the block structure we may write the homogeneous DAE as

$$\begin{bmatrix} X_1' \\ X_2' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -I_3 & 0 \\ \mathfrak{C} & \mathfrak{A} & \mathfrak{B} \\ -\mathfrak{B}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0, \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad X_3 = x_7. \quad (41)$$

Swapping the first two lines and also the variables leads to

$$\begin{bmatrix} X_2' \\ X_1' \\ 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{A} & \mathfrak{C} & \mathfrak{B} \\ -I_3 & 0 & 0 \\ 0 & -\mathfrak{B}^* & 0 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \\ X_3 \end{bmatrix} = 0, \quad \text{with } \mathfrak{B}^* \mathfrak{B} = 4\rho^2 > 0,$$

what shows Hessenberg structure of size three. The DAE (41) is regular with index $\mu = 3$ and features the sizes $m = 7$, $m_1 = 3$, $m_2 = 3$, $m_3 = 1$, dynamical degree of freedom $d = 4$, as well as characteristic values $r_0^T = r_1^T = r_2^T = 6$, $r_3^T = 7$.¹³ In terms of Definition 3.3 and (22) one has $r = 6$, $\theta_0 = \theta_1 = 1$, and $\theta_2 = 0$.

¹³See [18, Theorem 3.42]

The orthoprojector function $\Omega = \mathfrak{B}(\mathfrak{B}^*\mathfrak{B})^{-1}\mathfrak{B}^* = \frac{1}{4\rho^2}\mathfrak{B}\mathfrak{B}^*$ projecting pointwise the space \mathbb{R}^3 onto $\text{im } \mathfrak{B} \subset \mathbb{R}^3$ plays its role below. We have $\ker \Omega = \ker \mathfrak{B}^* = (\text{im } \mathfrak{B})^\perp$.

We choose a smooth basis $C_{\mathfrak{B}}$ of $\ker \mathfrak{B}^*$ such that

$$\begin{aligned} \text{im } C_{\mathfrak{B}} = \ker \mathfrak{B}^* &= (\text{im } \mathfrak{B})^\perp = \text{im}(I_3 - \Omega) \subset \mathbb{R}^3, \quad \text{rank } C_{\mathfrak{B}} = 2, \\ \text{im } \mathfrak{B} &= \text{im } \Omega, \quad \text{rank } \Omega = 1. \end{aligned} \quad (42)$$

It is evident that

$$S = \text{im} \begin{bmatrix} C_{\mathfrak{B}} & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \text{im} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \quad S \cap N = N, \quad \dim S \cap N = 1.$$

By calculating the general solution of the DAE (41) and inspecting the solution structure one obtains that

$$S_{can} = \text{im } C_{S_{can}} \subset S, \quad C_{S_{can}} = \begin{bmatrix} C_{\mathfrak{B}} & 0 \\ -\Omega' C_{\mathfrak{B}} & C_{\mathfrak{B}} \\ \mathfrak{K}_1 & \mathfrak{K}_2 \end{bmatrix},$$

with $\mathfrak{K}_1 = \frac{1}{4\rho^2}\mathfrak{B}^*(2\Omega' - \mathfrak{A})C_{\mathfrak{B}}$, $\mathfrak{K}_2 = \frac{1}{4\rho^2}\mathfrak{B}^*(\Omega'' - \mathfrak{C} + \mathfrak{A}\Omega' - 2\Omega'\Omega')C_{\mathfrak{B}}$.

7.1. An admissible matrix function sequence for DAE (41)

We start by

$$\Pi_0 = P_0 = G_0 = AD = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & -I_3 & 0 \\ \mathfrak{C} & \mathfrak{A} & \mathfrak{B} \\ -\mathfrak{B}^* & 0 & 0 \end{bmatrix}, \quad Q_0 = I_7 - P_0,$$

yielding

$$G_1 = G_0 + B_0 Q_0 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & \mathfrak{B} \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & -\frac{1}{4\rho^2}\mathfrak{B}^* & 0 \end{bmatrix},$$

and further

$$\begin{aligned} \Pi_0 Q_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 - \Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D\Pi_1 D^+ = \begin{bmatrix} I_3 & 0 \\ 0 & I_3 - \Omega \end{bmatrix}, \\ G_2 &= G_1 + B_0 \Pi_0 Q_1 - G_1 D^+ (D\Pi_1 D^+)' D \Pi_0 Q_1 \\ &= \begin{bmatrix} I_3 & -\Omega & 0 \\ 0 & I_3 + \mathfrak{A}\Omega + \Omega'\Omega & \mathfrak{B} \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Next we derive

$$Q_2 = \begin{bmatrix} \Omega & 0 & 0 \\ (I_3 - (I_3 - \Omega)(\mathfrak{A} + \Omega'))\Omega & 0 & 0 \\ \mathfrak{D} & 0 & 0 \end{bmatrix},$$

with $\mathfrak{D} = -\frac{1}{4\rho^2}\mathfrak{B}^*(I_3 + \mathfrak{A}\Omega + \Omega'\Omega)(I_3 - (I_3 - \Omega)\mathfrak{A}\Omega - \Omega'\Omega) = -\frac{1}{4\rho^2}\mathfrak{B}^*(I_3 + \mathfrak{A} + \Omega')\Omega$, and also

$$\Pi_1 Q_2 = \begin{bmatrix} \Omega & 0 & 0 \\ -(I_3 - \Omega)\mathfrak{A}\Omega - \Omega'\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I_3 - \Omega & 0 & 0 \\ (I_3 - \Omega)\mathfrak{A}\Omega + \Omega'\Omega & I_3 - \Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Regarding that $\text{im } G_0 = \text{im } G_1 = \text{im } G_2$ and owing to [18, Proposition 3.20] we determine the rank of $G_3 = G_2 + B_2 Q_2$ without knowing B_2 in detail:

$$\text{rank } G_3 = \text{rank } G_2 + \text{rank}(I_7 - G_0 G_0^+) B Q_2 = 6 + 1 = 7.$$

The projector function¹⁴ Π_2 is not identical with the canonical projector function Π_{can} since its image does not coincide with S_{can} . The determination of Π_{can} according to [18] is much more complex. But owing to [18, Theorem 2.8] which lists invariances of the construction of admissible sequences, it holds that

$$\begin{aligned} N_{can} &= \ker \Pi_{can} = \ker \Pi_2 \\ &= \left\{ \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : Z_1, Z_2 \in \text{im } \mathfrak{B}, Z_2 = Z_2 - (I_3 - \Omega) \mathfrak{A} Z_1 - \Omega' Z_1 \right\} \\ &= \text{im } C_{N_{can}}, \quad C_{N_{can}} = \begin{bmatrix} \mathfrak{B} & 0 & 0 \\ -((I_3 - \Omega) \mathfrak{A} + \Omega') \mathfrak{B} & \mathfrak{B} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (43)$$

7.2. Accurately stated initial conditions to (41) by using Π_2

The initial condition to (41),

$$G_a x(a) = \gamma, \quad (44)$$

with a matrix $G_a \in \mathbb{R}^{d \times m}$, is accurately stated, if

$$\text{im } G_a = \mathbb{R}^d, \quad \ker G_a = \ker \Pi_{can}.$$

We now intend to build a suitable matrix G_a . More precisely, we aim for a matrix function $G : \mathcal{I} \rightarrow \mathbb{R}^{d \times m}$, such that $G(a)$ may serve as G_a for each arbitrary $a \in \mathcal{I}$.

Introducing a matrix function $H : \mathcal{I} \rightarrow \mathbb{R}^{2 \times 3}$, with properties

$$\text{rank } H(t) = 2, \quad \ker H(t) = \text{im } \mathfrak{B}(t) = \text{im } \Omega(t), \quad t \in \mathcal{I},$$

we form

$$G = \begin{bmatrix} H & 0 & 0 \\ 0 & H & 0 \end{bmatrix} \Pi_2 = \begin{bmatrix} H & 0 & 0 \\ H(\mathfrak{A} + \Omega') \Omega & H & 0 \end{bmatrix}.$$

By construction, it holds that

$$\text{im } G(t) = \mathbb{R}^d, \quad \ker G(t) = \ker \Pi_2(t) = \ker \Pi_{can}(t), \quad t \in \mathcal{I}.$$

Below we choose¹⁵

$$H(t) = \begin{bmatrix} \sin t & -\cos t & 0 \\ 0 & 1 & \cos t \end{bmatrix}.$$

In particular, for $a = 0$, this yields

$$G(0) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

¹⁴We call attention to an error in the projector representation in [10, Section 6]. There the term $(I_3 - \Omega) \mathfrak{A} \Omega$ is missing. Instead of the above correct Π_2 there the incorrect version

$$\begin{bmatrix} I_3 - \Omega & 0 & 0 \\ \Omega' \Omega & I_3 - \Omega & 0 \\ 0 & 0 & 0 \end{bmatrix} =: \tilde{\Pi}_2.$$

is given.

¹⁵Note that $H = V^*$ can be chosen, with any basis V of $\ker \mathfrak{B}^*$.

7.3. Providing a matrix G_a such that $\ker G_a = N_{can}(a)$ via a basis of $S_{*can}(a)$

We write the homogeneous adjoint DAE to DAE (41) in the form

$$\begin{bmatrix} X_1' \\ X_2' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathfrak{C}^* & \mathfrak{B} \\ I & -\mathfrak{A}^* & 0 \\ 0 & -\mathfrak{B}^* & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0, \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, X_2 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}, X_3 = x_7, \quad (45)$$

and start the reduction of the pair of matrix functions featuring size 7×7 , $\text{rank } E_0 = 6$,

$$E_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & -\mathfrak{C}^* & \mathfrak{B} \\ I & -\mathfrak{A}^* & 0 \\ 0 & -\mathfrak{B}^* & 0 \end{bmatrix},$$

by choosing

$$Y_0^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad Z_0^* = [0 \quad 0 \quad 1],$$

and forming

$$S_0 = \ker Z^* F_0 = \left\{ \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : \mathfrak{B}^* Z_2 = 0 \right\}, \quad \dim S_0 = 6.$$

This leads to the following basis C_0 of the subspace S_0 ,

$$C_0 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & C_{\mathfrak{B}} & 0 \\ \underbrace{0}_3 & \underbrace{0}_2 & \underbrace{1}_1 \end{bmatrix},$$

and also to the reduced pair (size 6×6),

$$E_1 = Y_0^* E C_0 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & C_{\mathfrak{B}} & 0 \end{bmatrix}, \quad \text{rank } E_1 = 5,$$

$$F_1 = Y_0^* F C_0 + Y_0^* E C_0' = \begin{bmatrix} 0 & -\mathfrak{C}^* C_{\mathfrak{B}} & \mathfrak{B} \\ I_3 & -\mathfrak{A}^* C_{\mathfrak{B}} + C_{\mathfrak{B}}' & 0 \end{bmatrix},$$

Remember that $C_{\mathfrak{B}}$ is a smooth basis of $\ker \mathfrak{B}^*$ (42) and Ω is the orthoprojector onto $\text{im } \mathfrak{B}$. With

$$Y_1^* = \begin{bmatrix} I_3 & 0 \\ 0 & C_{\mathfrak{B}}^* \end{bmatrix}, \quad Z_1^* = [0 \quad \mathfrak{B}^*],$$

and

$$S_1 = \ker Z_1^* F_1 = \left\{ \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} : \mathfrak{B}^* Z_1 + \mathfrak{B}^* (C_{\mathfrak{B}}' - \mathfrak{A}^* C_{\mathfrak{B}}) Z_2 = 0 \right\},$$

$$\dim S_1 = 5,$$

we arrive at the smooth basis C_1 to the subspace S_1 ,

$$C_1 = \begin{bmatrix} C_{\mathfrak{B}} & -\Omega(C_{\mathfrak{B}}' - \mathfrak{A}^* C_{\mathfrak{B}}) & 0 \\ 0 & I_2 & 0 \\ \underbrace{0}_2 & \underbrace{0}_2 & \underbrace{1}_1 \end{bmatrix},$$

as well as the next reduced pair, (size 5×5),

$$E_2 = Y_1^* E_1 C_1 = \begin{bmatrix} C_{\mathfrak{B}} & -\mathfrak{F} & 0 \\ 0 & C_{\mathfrak{B}}^* C_{\mathfrak{B}} & 0 \end{bmatrix}, \quad \mathfrak{F} = \Omega(C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}) \quad \text{rank } E_2 = 4,$$

$$F_2 = Y_1^* F_1 C_1 + Y_1^* E_1 C'_1 = \begin{bmatrix} C'_{\mathfrak{B}} & -\mathfrak{E}^* C_{\mathfrak{B}} - \mathfrak{F}' & \mathfrak{B} \\ C_{\mathfrak{B}}^* C_{\mathfrak{B}} & C_{\mathfrak{B}}^* (C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}) & 0 \end{bmatrix},$$

With

$$Z_2^* = [\mathfrak{B}^* \quad \mathfrak{B}^* \mathfrak{E}], \quad \mathfrak{E} = \Omega \mathfrak{F} (C_{\mathfrak{B}}^* C_{\mathfrak{B}})^{-1},$$

we obtain

$$S_2 = \ker Z_2^* F_2 = \left\{ \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} : -\mathfrak{H}_1 Z_1 - \mathfrak{H}_2 Z_2 + Z_3 = 0 \right\},$$

$$\mathfrak{H}_1 = -\frac{1}{4\rho^2} \mathfrak{B}^* (\Omega C'_{\mathfrak{B}} + \mathfrak{F}),$$

$$\mathfrak{H}_2 = -\frac{1}{4\rho^2} \mathfrak{B}^* (-\Omega (\mathfrak{E}^* C_{\mathfrak{B}} + \mathfrak{F}') + \mathfrak{E} C_{\mathfrak{B}}^* (C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}))$$

$$\dim S_2 = 4,$$

as well as the basis

$$C_2 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \\ \underbrace{\mathfrak{H}_1}_2 & \underbrace{\mathfrak{H}_2}_2 \end{bmatrix},$$

Finally, taking

$$Y_2^* = \begin{bmatrix} C_{\mathfrak{B}}^* & 0 \\ 0 & I_2 \end{bmatrix},$$

the next matrix function E_3 (size 4×4) remains nonsingular, namely

$$E_3 = Y_2^* E_2 C_2 = \begin{bmatrix} C_{\mathfrak{B}}^* C_{\mathfrak{B}} & -C_{\mathfrak{B}}^* \mathfrak{F} \\ 0 & C_{\mathfrak{B}}^* C_{\mathfrak{B}} \end{bmatrix},$$

and, hence the matrix function $C = C_0 C_1 C_2$ showing size 7×4 ,

$$C = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & C_{\mathfrak{B}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{\mathfrak{B}} & -\Omega(C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}) & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \\ \underbrace{\mathfrak{H}_1}_2 & \underbrace{\mathfrak{H}_2}_2 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\mathfrak{B}} & -\Omega(C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}) \\ 0 & C_{\mathfrak{B}} \\ \mathfrak{H}_1 & \mathfrak{H}_2 \end{bmatrix}$$

serves as basis of S_{*can} ¹⁶. We are mainly interested in the matrix function

$$C^* E_0 = \begin{bmatrix} C_{\mathfrak{B}}^* & 0 & \mathfrak{H}_1^* \\ (-\Omega(C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}))^* & C_{\mathfrak{B}}^* & \mathfrak{H}_2^* \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\mathfrak{B}}^* & 0 & 0 \\ (-\Omega(C'_{\mathfrak{B}} - \mathfrak{A}^* C_{\mathfrak{B}}))^* & C_{\mathfrak{B}}^* & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\mathfrak{B}}^* & 0 & 0 \\ ((\Omega' + \mathfrak{A}^*) C_{\mathfrak{B}})^* & C_{\mathfrak{B}}^* & 0 \end{bmatrix} = \begin{bmatrix} C_{\mathfrak{B}}^* (\Omega' + \mathfrak{A}) & 0 & 0 \\ C_{\mathfrak{B}}^* & 0 & 0 \end{bmatrix}$$

¹⁶Note that (45) is the adjoint to (41).

and its three-dimensional nullspace

$$\ker C^*E = \left\{ \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : Z_2 = Z_{2b} - (I - \Omega)(\Omega' + \mathfrak{A})Z_1, Z_1, Z_{2b} \in \text{im } \mathfrak{B} \right\}$$

which coincides with the subspace N_{can} in (43).

7.4. Canonical projector function to (41)

Having the bases functions $C_{S_{can}}, C_{N_{can}}$ of S_{can}, N_{can} , respectively, the 7×7 matrix function $\mathcal{M} := [C_{S_{can}} C_{N_{can}}]$ remains nonsingular everywhere on \mathcal{I} , and

$$\Pi_{can} = \mathcal{M} \begin{bmatrix} I_4 & \\ & 0_3 \end{bmatrix} \mathcal{M}^{-1}.$$

8. Conclusion, discussion and outlook

Based on ideas in [25] we have proposed an easily accessible notion of regularity together with characteristic values and an index for linear time-varying DAEs. Moreover, we have shown the close relation to other index notions and their characteristic values by comparing it with the other concepts in [5, 16, 17, 18]. This enables the simultaneous use of the corresponding theories and their results.

We have shed light on the fact that, for any regular DAE acting in \mathbb{R}^m , two important time-varying canonical subspaces are well-defined, namely the *flow-subspace* S_{can} and *its canonical complement* N_{can} yielding the pointwise decomposition $S_{can}(t) \oplus N_{can}(t) = \mathbb{R}^m$. Furthermore, a relatively simple procedure for the determination of *accurate initial conditions* could be established by exploiting the connection of the canonical subspaces of a DAE and its adjoint. In particular, this also results in an accurate transfer condition for window-by-window forward least-squares collocation, which previously had to be prepared manually. We will devote another project to the detailed analysis of numerical feasibility.

Especially the investigations in Section 4 concerning different concepts to treat DAEs can certainly be seen as a step towards an comprehensive survey, and one may ask why other important approaches were not included. The answer is that we have included only those concepts that we needed for our goal, our reasoning and proofs, that is, those concepts which generalize the Kronecker index, do not use any derivative arrays, and whose characteristics persist under equivalence transformations. This excludes the structural index, but also all versions of the differentiation index. Nevertheless a project that also incorporates approaches based on derivative arrays is being worked on.

A. Appendix

A.1. Standard form DAEs, DAEs with properly involved derivative, and proper factorizations

We say that N is a C^1 -subspace in \mathbb{R}^n , if $N(t) \subseteq \mathbb{R}^n$ is a time-varying subspace, $t \in \mathcal{I}$, and the projector-valued function $Q : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$, with $Q(t) = Q(t)^2 = Q(t)^T$, $\text{im } Q(t) = N(t)$, $t \in \mathcal{I}$, is continuously differentiable. We underline that any C^1 -subspace in \mathbb{R}^n has constant dimension and also a continuously differentiable basis.

Each continuous matrix function $E : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ having constant rank r and a nullspace which is a C^1 -subspace in \mathbb{R}^m can be factorized into $E = AD$, with $A : \mathcal{I} \rightarrow \mathbb{R}^{m \times n}$, $D : \mathcal{I} \rightarrow \mathbb{R}^{n \times m}$, so that A is continuous, D is continuously differentiable, $\ker A$ and $\text{im } D$ are a C^1 -subspaces in \mathbb{R}^n , and

$$\ker A(t) \oplus \text{im } D(t) = \mathbb{R}^n, \quad t \in \mathcal{I}. \quad (46)$$

A possible choice is $n = m$, $A = E$, $D = E^+E$ with the pointwise Moore-Penrose generalized inverse E^+ of E . If E is itself continuously differentiable, then also the factor A can be chosen to be continuously differentiable, for instance $A = EE^+$, $D = E$. Any factorization satisfying condition (46) is called *proper factorization*. Note that then the function $R : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$, projecting pointwise onto $\text{im} D$ along $\text{ker} A$ is also continuously differentiable and one has

$$\text{im} E = \text{im} A, \text{ker} E = \text{ker} D, \quad A = AR, D = RD. \quad (47)$$

R is then called *border-projector function*.

Using any proper factorization of the leading coefficient E , the standard form DAE $Ex' + Fx = q$ can be rewritten with $B = F - AD'$ as DAE with *properly stated leading term or DAE with properly involved derivative*, $A(Dx)' + Bx = q$.

Of course, on the other hand, starting from a DAE with properly involved derivative, $A(Dx)' + Bx = q$, one immediately gains the DAE in standard form $ADx' + (B + AD')x = q$.

A properly involved derivative is essential when rigorous solvability statements are required and only the component Dx can be expected to be continuously differentiable. In the case of DAEs with proper involved derivative, the DAE and its adjoint show a formal symmetry that proves beneficial for the analysis. However, if, as here, we are concerned with the description of the time-varying subspaces in \mathbb{R}^m that capture the solution values, then the form of the inclusion of the derivative does not matter, in particular,

$$\begin{aligned} N &= \text{ker} E = \text{ker} AD = \text{ker} D, \\ S &= \{z \in \mathbb{R}^m : Fz \in \text{im} E\} = \{z \in \mathbb{R}^m : Bz \in \text{im} AD\}, \end{aligned}$$

and both DAE forms share their canonical projector Π_{can} and the related canonical subspaces S_{can} and N_{can} .

A.2. Modification of the reduction procedure for DAEs (2)

We sketch here a modification of the reduction procedure from [25] for DAEs with properly involved derivative (2).

We start by $A_0 = A$, $D_0 = D$, $B_0 = B$, $m_0 = m$, $r_0 = r$ and consider the homogeneous DAE

$$A_0(D_0x)' + B_0x = 0.$$

By means of a basis $Z_0 : \mathcal{I} \rightarrow \mathbb{R}^{(m_0-r_0) \times m_0}$ of $(\text{im} A_0)^\perp = \text{ker} A_0^*$ and a basis $Y_0 : \mathcal{I} \rightarrow \mathbb{R}^{r_0 \times m_0}$ of $\text{im} A_0$ we divide the DAE into the two parts

$$Y_0^* A_0(D_0x)' + Y_0^* B_0x = 0, \quad Z_0^* B_0x = 0.$$

From $\text{im}[A_0 D_0 \ B_0] = \mathbb{R}^m$ we derive that $\text{rank} Z_0^* B = m_0 - r_0$, and hence. the subspace $S_0 = \text{ker} Z_0^* B = \text{ker} Z_0^* F$ has dimension r_0 . Each solution of the homogeneous DAE must stay in the subspace S_0 . Choosing a basis $C_0 : \mathcal{I} \rightarrow \mathbb{R}^{r_0 \times m_0}$ of S_0 , each solution of the DAE can be represented as $x = C_0 x_{[1]}$, with a function $x_{[1]} : \mathcal{I} \rightarrow \mathbb{R}^{r_0}$ satisfying the reduced to size $m_1 = r_0$ DAE given below. In contrast to [25] where the basis C_0 is required to be continuously differentiable, we suppose now a continuous basis C_0 which has a continuously differentiable part $D_0 C_0$. Using a pointwise generalized inverse $(D_0 C_0)^-$ of $D_0 C_0$ we may write

$$\begin{aligned} D_0 x &= D_0 C_0 x_{[1]} = D_0 C_0 \underbrace{(D_0 C_0)^- D_0 C_0}_{=: D_1} x_{[1]}, \\ (D_0 x)' &= D_0 C_0 (D_1 x_{[1]})' + (D_0 C_0)' D_1 x_{[1]}. \end{aligned}$$

This leads to a DAE living in \mathbb{R}^{m_1} , $m_1 := r_0$, with properly involved derivative,

$$\underbrace{Y_0^* A_0 D_0 C_0 (D_1 x_{[1]})'}_{=:A_1} + \underbrace{Y_0^* (B_0 C_0 + A_0 (D_0 C_0)' D_1)}_{=:B_1} x_{[1]} = 0.$$

A.3. Basic steps by Chistyakov and Jansen

Let the pair $\{E, F\}$, $E, F : \mathcal{S} \rightarrow \mathbb{R}^{m \times m}$, be pre-regular with constants r and θ according to Definition 3.1. We take over some notations from Section 4.

Let T, T^c, Z , and Y represent bases of $\ker E$, $(\ker E)^\perp$, $(\text{im } E)^\perp$, and $\text{im } E$, respectively.

The matrix function $Z^* F T$ has size $(m-r) \times (m-r)$, rank $m-r-\theta = a$, and $\ker Z^* F T = T^+(N \cap S)$ has dimension θ . By scaling with $[Y Z]^*$ one splits the DAE

$$E x' + F x = q$$

into the partitioned shape

$$Y^* E x' + Y^* F x = Y^* q, \quad (48)$$

$$Z^* F x = Z^* q. \quad (49)$$

Owing to the pre-regularity, the $(m-r) \times m$ matrix function $Z^* F$ features full row-rank $m-r$. We keep in mind that $S = \ker Z^* F$ has dimension r .

A.3.1. Elimination by Chistyakov

Taking a nonsingular matrix function K of size $m \times m$ such that $Z^* F K =: [\tilde{F}_{21} \ \tilde{F}_{22}]$, with \tilde{F}_{22} being nonsingular, the transformation $x = K \tilde{x}$ turns (49) into

$$\begin{aligned} Z^* F x = Z^* F K \tilde{x} &=: \tilde{F}_{21} \tilde{x}_1 + \tilde{F}_{22} \tilde{x}_2 = Z^* q, \\ \text{yielding } \tilde{x}_2 &= -\tilde{F}_{22}^{-1} \tilde{F}_{21} \tilde{x}_1 + Z^* q. \end{aligned}$$

Next one eliminates the variable \tilde{x}_2 in the transformed version of (48),

$$\underbrace{Y^* E K}_{=: [\tilde{E}_{11} \ \tilde{E}_{12}]} \tilde{x}' + (Y^* F K + Y^* E K') \tilde{x} = Y^* q,$$

which yields a DAE for $\tilde{x}_1 =: x_{[1]}$,

$$\underbrace{(\tilde{E}_{11} - \tilde{E}_{12} \tilde{F}_{22}^{-1} \tilde{F}_{21})}_{=: E_{[1]}} \tilde{x}'_1 + F_{[1]} \tilde{x}_1 = Y^* q.$$

A further look at the matter shows that we are dealing with a special basis of the subspace S , namely

$$S = \text{im } C, \quad C = K \begin{bmatrix} I_r \\ -\tilde{F}_{22}^{-1} \tilde{F}_{21} \end{bmatrix},$$

whereby this early predecessor of the method described in [25] can now be classified as its special version. Note that the procedure in [25] allows for the choice of an arbitrary basis for S .

It should be further mentioned, that in [5] the elimination method is applied not only for pre-regular pairs but for general rectangular matrix functions E, F .

A.3.2. Dissection by Jansen

The approach in [16] needs several more splittings. As before let T, T^c, Z , and Y represent bases of $\ker E, (\ker E)^\perp, (\operatorname{im} E)^\perp$, and $\operatorname{im} E$, respectively. Additionally, let V, W be bases of $(\operatorname{im} Z^*FT)^\perp$, and $\operatorname{im} Z^*FT$. By construction, V has size $(m-r) \times a$ and W has size $(m-r) \times \theta$. One starts with the transformation

$$x = \begin{bmatrix} T^c & T \end{bmatrix} \tilde{x}, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \quad x = T^c \tilde{x}_1 + T \tilde{x}_2.$$

The background is the associated possibility to suppress the derivative of the nullspace-part $T\tilde{x}_n$ similarly as in the context of properly formulated DAEs and to set $Ex' = ET^c\tilde{x}'_1 + ET^{c'}\tilde{x}_1 + ET'\tilde{x}_2$, which, however, does not play a role here where altogether continuously differentiable solutions are assumed. Furthermore, an additional partition of the derivative-free equation (49) by means of the scaling with $[VW]^*$ is applied, which results in the system

$$Y^*ET^c\tilde{x}'_1 + Y^*(FT^c + ET^{c'})\tilde{x}_1 + Y^*(FT + ET')\tilde{x}_2 = Y^*q, \quad (50)$$

$$V^*Z^*FT^c\tilde{x}_1 + V^*Z^*FT\tilde{x}_2 = V^*Z^*q, \quad (51)$$

$$W^*Z^*FT^c\tilde{x}_1 = W^*Z^*q. \quad (52)$$

The matrix function $W^*Z^*FT^c$ has full row-rank θ and V^*Z^*FT has full row-rank a . Now comes another split. Choosing bases G, H of $\ker W^*Z^*FT^c \subset \mathbb{R}^\theta$ and $\ker V^*Z^*FT \subset \mathbb{R}^a$, as well as bases of respective complementary subspaces, we transform

$$\tilde{x}_1 = \begin{bmatrix} G^c & G \end{bmatrix} \bar{x}_1 \quad \bar{x}_1 = \begin{bmatrix} \bar{x}_{1,1} \\ \bar{x}_{1,2} \end{bmatrix}, \quad \tilde{x}_1 = G^c \bar{x}_{1,1} + G \bar{x}_{1,2},$$

$$\tilde{x}_2 = \begin{bmatrix} H^c & H \end{bmatrix} \bar{x}_2 \quad \bar{x}_2 = \begin{bmatrix} \bar{x}_{2,1} \\ \bar{x}_{2,2} \end{bmatrix}, \quad \tilde{x}_2 = H^c \bar{x}_{2,1} + H \bar{x}_{2,2}.$$

Thus equations (51) and (52) are split into

$$\begin{aligned} V^*Z^*FT^c(G^c \bar{x}_{1,1} + G \bar{x}_{1,2}) + V^*Z^*FTH^c \bar{x}_{2,1} &= V^*Z^*q, \\ W^*Z^*FT^c G^c \bar{x}_{1,1} &= W^*Z^*q. \end{aligned}$$

The matrix functions $V^*Z^*FTH^c$ and $W^*Z^*FT^c G^c$ are nonsingular each, which allows the resolution to $\bar{x}_{1,1}$ and $\bar{x}_{2,1}$. In particular, for $q = 0$ it results that $\bar{x}_{1,1} = 0$ and $\bar{x}_{2,1} = \mathfrak{E} \bar{x}_{1,2}$, with

$$\mathfrak{E} := -(V^*Z^*FTH^c)^{-1} V^*Z^*FT^c G^c.$$

Overall, therefore, the latter procedure presents again a transformation, namely

$$x = K\bar{x}, \quad K = \begin{bmatrix} T^c G^c & T^c G & TH^c & TH \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \bar{x}_{1,1} \\ \bar{x}_{1,2} \\ \bar{x}_{2,1} \\ \bar{x}_{2,2} \end{bmatrix} \in \mathbb{R}^\theta \times \mathbb{R}^{r-\theta} \times \mathbb{R}^a \times \mathbb{R}^\theta,$$

and we realize that we have found again a basis of the subspace S , namely

$$S = \operatorname{im} C, \quad C = K \begin{bmatrix} 0 & 0 \\ I_{r-\theta} & 0 \\ \mathfrak{E} & 0 \\ 0 & I_\theta \end{bmatrix} = \begin{bmatrix} T^c G + TH^c \mathfrak{E} & TH \end{bmatrix},$$

which makes the dissection approach a special case of [25]. The characteristic values together with the index are formally adapted to the values of the tractability index. It starts with $r_0^D = r$, and is continued in ascending order with $r_{i+1}^D = r_i^D + a_i = r_i^D + \operatorname{rank} Z_i^+ F_i T_i$ etc. until $r_{\mu-1}^D < r_\mu^D = m$, [16, Definition 4.13].

It should be noted, however, that the dissection concept is developed with considerable effort for nonlinear DAEs in [16].

A.4. Admissible matrix function sequences and related subspaces

In this part we apply several routine notations and tools used in the projector based analysis of DAEs. We refer to the appendix for a short roundup and to [18, 24] for more details.

Given are at least continuous matrix functions $E, F : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$, E has a C^1 -nullspace and constant rank r . We use a proper factorization $E = AD$ where $A : \mathcal{I} \rightarrow \mathbb{R}^{m \times k}$, $D : \mathcal{I} \rightarrow \mathbb{R}^{k \times m}$, and $B = -(F + AD')$. $R : \mathcal{I} \rightarrow \mathbb{R}^{k \times k}$ denotes the continuously differentiable projector-valued function such that $\text{im} D = \text{im} R$ and $\ker A = \ker R$.

Let $Q_0 : \mathcal{I} \rightarrow \mathbb{R}^{m \times m}$ denote any continuously differentiable projector-valued function such that $\text{im} Q_0 = \ker D = \ker E$, for instance, $Q_0 = I - D^+ D$ with the pointwise Moore-Penrose inverse D^+ . Set $P_0 = I - Q_0$ and let D^- denote the pointwise generalized inverse of D determined by

$$D^- D D^- = D^-, \quad D D^- D = D, \quad D D^- = R, \quad D^- D = P_0.$$

Set $G_0 = AD$, $B_0 = B$, $\Pi_0 = P_0$. For a given level $\kappa \in \mathbb{N}$, the sequence G_0, \dots, G_κ is called an *admissible matrix function sequence* associated with the pair $\{E, F\}$ and triple $\{A, D, B\}$, respectively, e.g., [18, Definition 2.6] if it is built by the rule

$$\begin{aligned} G_i &= G_{i-1} + B_{i-1} Q_{i-1}, \\ B_i &= B_{i-1} P_{i-1} - G_i D^- (D \Pi_i D^-)' D \Pi_{i-1}, \\ N_i &= \ker G_i, \quad \widehat{N}_i := (N_0 + \dots + N_{i-1}) \cap N_i, \quad N_0 + \dots + N_{i-1} =: \widehat{N}_i \oplus X_i, \\ &\text{choose } Q_i \text{ such that } Q_i = Q_i^2, \text{ im } Q_i = N_i, X_i \subseteq \ker Q_i, \\ P_i &= I - Q_i, \quad \Pi_i = \Pi_{i-1} P_i, \\ i &= 1, \dots, \kappa, \end{aligned}$$

and, additionally,

- (a) G_i has constant rank r_i^T , $i = 0, \dots, \kappa$,
- (b) $\widehat{N}_i = (\text{im}[\Pi_{i-1}^* G_i^*])^\perp$ has constant dimension u_i^T , $i = 1, \dots, \kappa$,
- (c) Π_i is continuous and $D \Pi_i D^-$ is continuously differentiable, $i = 0, \dots, \kappa$.

The admissible matrix functions G_i are continuous. The construction is supported by two constant-rank conditions at each level. It results that

$$\begin{aligned} 0 &< r_0^T \leq r_1^T \leq \dots \leq r_i^T \leq \dots, \\ 0 &\leq u_1^T \leq u_2^T \leq \dots \leq u_i^T \leq \dots \end{aligned}$$

By construction, the inclusions

$$\text{im } G_0 \subseteq \text{im } G_1 \subseteq \dots \subseteq \text{im } G_{r+1} = \text{im } G_{r+2} \tag{53}$$

are valid pointwise. There are several special projector functions incorporated in an admissible matrix function sequence, among them admissible projectors Q_i onto $\ker G_i$ and $\Pi_i = \Pi_{i-1}(I - Q_i)$, $\Pi_0 = (I - Q_0)$, yielding the further inclusions

$$\ker \Pi_0 \subseteq \ker \Pi_1 \subseteq \dots \subseteq \ker \Pi_r = \ker \Pi_{r+1}. \tag{54}$$

Each of the time-varying subspaces in (53) and (54) has constant dimension, which is ensured by the respective rank conditions.

The subspaces involved in (53) and (54) are proved to be invariant with respect to special possible choices within the construction procedure and also with respect to the factorization of $E = AD$.

Definition A.1. The DAE given by the coefficient function pair $\{E, F\}$ or a related triple $\{A, D, B\}$ is called *regular* if there are an index μ and an admissible matrix function sequence G_0, G_1, \dots, G_μ such that $r_\mu^T = m$. The *tractability index* of the DAE is defined to be the smallest index μ with $r_\mu^T = m$, and the integers

$$0 < r_0^T \leq r_1^T \leq \dots \leq r_{\mu-1}^T < r_\mu^T = m$$

are called *characteristic values* of the regular DAE.

For a regular DAE $N_\mu = \{0\}$ is trivially valid, therefore $\widehat{N}_\mu = \{0\}$ must also be valid and in turn $u_i^T = 0$, for all i . Moreover, owing to [18, Proposition 2.5] regularity requires

$$\text{im}[E \ F] = \mathbb{R}^m, \quad \text{im}[A \ D \ B] = \mathbb{R}^m.$$

Data Availability Statement

Not applicable.

Underlying and related material

Not applicable.

Author contributions

We follow the good tradition in mathematical research, emphasizing content and interaction, of arranging the participating authors alphabetically and assuming that all authors are equally involved and responsible.

Competing interests

None.

References

- [1] T. Berger, A. Ilchmann, S. Trenn. The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl.* 2012;436:4052–4069.
- [2] K. E. Brenan, S. L. Campbell, L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. North-Holland, Elsevier Science Publishing Co., Inc. Amsterdam, 1989.
- [3] M. Burger, M. Gerds. “Surveys in differential-algebraic equations IV”. *Differential-Algebraic Equations Forum*. eds. Achim Ilchmann and Timo Reis. Springer Heidelberg, 2017. Chap. A Survey on Numerical Methods for the Simulation of Initial Value Problems with sDAEs:221–300.
- [4] S. Campbell, E. Moore. Constraint preserving integrators for general nonlinear higher index DAEs. *Numer Math.* 1995;69:383–399.
- [5] V. F. Chistyakov. “K metodam resheniya singulyarnykh lineynykh sistem obyknovennykh differentsial’nykh uravnenij”. *Vyrozhdennye sistemy obyknovennykh differentsial’nykh uravnenij*. Ed. by Y. E. Boyarintsev. In Russian. Nauka, Sibirskoe Otdelenie, 1982:37–65.
- [6] D. Estévez Schwarz, R. Lamour. A new approach for computing consistent initial values and taylor coefficients for daes using projector-based constrained optimization. *Numer. Algorithms.* 2018;78(2):355–377.

- [7] E. Griepentrog, R. März. *Differential-Algebraic Equations and Their Numerical Treatment*. Teubner Texte zur Mathematik 88. BSB Teubner Leipzig, 1986.
- [8] M. Hanke, R. März. A reliable direct numerical treatment of differential-algebraic equations by overdetermined collocation: an operator approach. *J Comp Appl Math.* 2021;387.
- [9] M. Hanke, R. März. Convergence analysis of least-squares collocation methods for nonlinear higher-index differential-algebraic equations. *J Comp Appl Math.* 2021;387.
- [10] M. Hanke, R. März. Least-squares collocation for higher-index daes: global approach and attempts toward a time-stepping version. *Progress in Differential-Algebraic Equations II*. Ed. by T Reis, S. Grundel, S. Schöps. Differential-Algebraic Equations Forum. Cham: Springer, 2020:91–135.
- [11] M. Hanke, R. März, C. Tischendorf. Least-squares collocation for higher-index linear differential-algebraic equations: estimating the stability threshold. *Math. Comp.* 2019;88(318):1647–1683.
- [12] M. Hanke, R. März, C. Tischendorf, E. Weinmüller, S. Wurm. Least-squares collocation for linear higher-index differential-algebraic equations. *J. Comput. Appl. Math.* 2017;317:403–431.
- [13] M. Hanke. On the sensitivity of implementation of a least-squares collocation method for linear higher-index differential-algebraic equations. *Numerical Algorithms.* 2022;91:1721–1754.
- [14] M. Hanke, R. März. Towards a reliable implementation of least-squares collocation for higher-index differential-algebraic equations –part 1: basics and ansatz function choices. *Numerical Algorithms.* 2022;89:931–963.
- [15] M. Hanke, R. März. Towards a reliable implementation of least-squares collocation for higher-index differential-algebraic equations –part 2: the discrete least-squares problem. *Numerical Algorithms.* 2022;89:965–986.
- [16] L. Jansen. “A dissection concept for daes. structural decoupling, unique solvability, convergence theory and half-explicit methods”. PhD thesis. Humboldt-Universität zu Berlin, 2014.
- [17] P. Kunkel, V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. Textbooks in Mathematics. Zürich: European Mathematical Society, 2006.
- [18] R. Lamour, R. März, C. Tischendorf. *Differential-Algebraic Equations: A Projector Based Analysis*. Differential-Algebraic Equations Forum. eds. Achim Ilchmann and Timo Reis. Springer-Verlag Berlin Heidelberg New York Dordrecht London, 2013.
- [19] R. Lamour, R. März, E. Weinmüller. “Surveys in differential-algebraic equations III”. *Differential-Algebraic Equations Forum*. eds. Achim Ilchmann and Timo Reis. Springer Heidelberg, 2015. Chap. Boundary-Value Problems for Differential-Algebraic Equations: A Survey:177–309.
- [20] V. H. Linh, R. März. Adjoint pairs of differential-algebraic equations and their Lyapunov exponents. *J Dyn Diff Equat.* 2017;29(2):655–684.
- [21] V. H. Linh, V. Mehrmann. “Control and optimization with differential-algebraic constraints”. SIAM Philadelphia, 2012. Chap. Spectra and Leading Directions for Linear DAEs:59–78.
- [22] R. März. Canonical projectors for linear differential-algebraic equations. *Comput. Math. Appl.* 1996;31(4-5):121–135.
- [23] R. März. New answers to an old question in the theory of differential-algebraic equations: essential underlying ODE versus inherent ODE. *J. Comput. Appl. Math.* 2017;316:271–286.
- [24] R. März. “Surveys in differential-algebraic equations II”. *Differential-Algebraic Equations Forum*. eds. Achim Ilchmann and Timo Reis. Springer Heidelberg, 2015. Chap. Differential-Algebraic Equations from a Functional-Analytic Viewpoint: A Survey:163–285.
- [25] P. J. Rabier, W. C. Rheinboldt. “Handbook of numerical analysis, vol. VIII”. Elsevier Science, 2002. Chap. Theoretical and Numerical Analysis of Differential-Algebraic Equations.
- [26] R. Riaza. *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications*. World Scientific, 2008.

- [27] S. Trenn. “Surveys in differential-algebraic equations I”. Differential-Algebraic Equations Forum. eds. Achim Ilchmann and Timo Reis. Springer Heidelberg, 2013. Chap. Solution Concepts for Linear DAEs: A Survey:137–172.
- [28] K.-T. Wong. The eigenvalue problem $\lambda Tx + Sx$. *J Differential Equations*. 1974;16(2):270–280.
- [29] N. Wong. An efficient passivity test for descriptor systems via canonical projector techniques. Proceedings of the 46th Annual Design Automation Conference, DAC '09. ACM, New York. 2009.

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